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The theory and practice of  
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THE  
THEORY AND PRACTICE  
OF  
INTERPOLATION:

INCLUDING

MECHANICAL QUADRATURE, AND OTHER IMPORTANT PROBLEMS  
CONCERNED WITH THE TABULAR VALUES OF FUNCTIONS.

WITH THE REQUISITE TABLES.

BY

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## PREFACE.

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IN preparing the following treatise the author has attempted no marked originality, either of subject matter or method. Indeed, sufficient has hitherto been written of Interpolation, Quadratures, etc., to firmly dissuade one from such an endeavor. Yet of the numerous contributions to these allied subjects, there has appeared thus far no distinct treatise covering the entire ground. As a consequence the author has repeatedly felt the need of a work which would give—exclusive of other matter—a simple, practical, yet comprehensive discussion of all that is useful concerning Differences, Interpolation, Tabular Differentiation and Mechanical Quadrature;—a work, moreover, which would include all tables appertaining to the text which are required by a practical computer. To supply the want thus conceived, the author offers the present volume.

But while viewing the matter in this practical sense, the writer regards his work as no mere compilation. Many of the processes and developments are original, so far as he is concerned, and possibly altogether new; while the same remark applies to a few of the minor *results*. In fact, if adverse criticism be forthcoming, it will probably result largely from the somewhat unusual or individual methods which in many instances have been employed in preference to the customary forms of analysis. On the other hand the author realizes fully the extent of his indebtedness to previous writers for valuable ideas and suggestions; and he desires especially to mention the works of BOOLE, CHAUVENET, ENCKE, LOOMIS, NEWCOMB, and SAWITSCH as most valuable sources of information, to which frequent reference has been made.

Concerning the bibliographical list at the close of this volume (which includes the foregoing names), it is but proper to state that references to several of the earliest writers—such as BRIGGS, WALLIS, MOUTON, COTES, STIRLING, MAYER, WALMESLEY, LALANDE—have purposely been omitted because of the general inaccessibility of their works. As regards the writings of the present century, however, the author believes that all contributions of importance have been included, and trusts that any omissions of consequence hereafter detected will be regarded merely as oversights.

Special care has been given to the preparation and printing of the tables, with the hope of securing absolute accuracy. At a considerable cost of labor, and by wholly independent methods, the computations were all made in duplicate; and in every case the tabular values are true to the *nearest* unit of the last place. Though a few of these tables have appeared before, several are here published for the first time, and it is hoped they will prove useful to the computer.

In conclusion, the author desires to express his cordial thanks and appreciation to Mr. E. C. RUEBSAM, of the Nautical Almanac Office, and to Mr. M. E. PORTER, of the Naval Observatory, for much valuable service and many useful suggestions received during the various phases of preparation of this treatise. Feelings of gratitude further inspire—simple justice even demands—a special word in commendation of the publishers, whose uniform courtesy, accuracy and skill have done much to enhance the general value of the work.

H. L. R.

WASHINGTON, D.C., *December*, 1899.

# CONTENTS.

## CHAPTER I.

### OF DIFFERENCES.

Section.		Page.
1.	General remarks concerning tabular functions and the construction of mathematical tables, . . . . .	1
2.	Fundamental definitions and notation. General schedule of functions and differences, . . . . .	2
3.	Method of checking the numerical accuracy of differences. Theorem I,	4
4.	$N$ functions yield $N-n$ $n$ th differences. Theorem II, . . . .	5
5.	Effect of inverting a given series. Theorem III, . . . . .	5
6.	Differences of two combined series. Theorem IV, . . . . .	6
7.	Irregularities in the differences of functions which are not mathematically exact, . . . . .	7
8.	Detection of accidental errors by differences, . . . . .	9
9.	Numerical examples—in which only one function requires correction,	11
10.	Numerical examples—involving two or three erroneous functions, .	13
11.	General properties of differences. Expression of $A_0^{(n)}$ in terms of the $n$ th and higher derivatives of $F(t)$ ;—equation (4), . . . .	15
12.	Determination of the coefficients $B, C, D$ , etc., in equation (4), . . .	18
13.	Remarkable formal relation between the expressions for $A_0^{(n)}$ and $A_0'$ , .	21
14.	The $n$ th differences of any rational integral function of the $n$ th degree are constant. Theorem V, . . . . .	23
15.	Converse of the foregoing proposition. Theorem VI, . . . . .	24
16, 17.	Convergency of differences. <i>Magnitude of tabular interval and character of function</i> the principal elements involved. Numerical illustrations, . . . . .	25
18.	Expression of $\omega F'(t)$ , $\omega^2 F''(t)$ , etc., in terms of tabular differences, .	28
19.	Change of the argument interval from $\omega$ to $m\omega$ : effect upon the magnitude of the successive differences, . . . . .	30
20.	Practical result of the foregoing investigation. Theorem VII, . . .	34
21.	Numerical example—reduction of tabular interval, . . . . .	34
22.	Expression of any difference in terms of tabular functions, . . . .	35
23.	Expression of any tabular function in terms of $F_0, A_0', A_0'', A_0'''$ , etc., Examples, . . . . .	36

## CHAPTER II.

## OF INTERPOLATION.

Section.		Page.
24.	Statement of the problem, . . . . .	40
25.	Rigorous proof of NEWTON'S Formula, assuming that differences of some particular order are constant, . . . . .	41
26.	Second demonstration of NEWTON'S Formula, restricted as in §25, . .	43
27.	Formula for computing the interval $n$ , . . . . .	43
28.	Example of interpolation by NEWTON'S Formula, the fourth differences being constant, . . . . .	44
29.	<i>Backward</i> interpolation by NEWTON'S Formula. Interpolation near the <i>end</i> of a series. Numerical example, . . . . .	44
30.	General investigation proving that NEWTON'S Formula is sensibly accu- rate as applied to series whose differences practically — though not absolutely — vanish beyond the 4th or 5th order, . . . . .	46
31.	Numerical example illustrating the foregoing discussion, . . . .	57
32, 33.	Practical examples in the use of NEWTON'S Formula, . . . . .	61
34.	Transformations of NEWTON'S Formula. Modification of the foregoing notation of differences. STIRLING'S Formula. Schedule of differ- ences referring to same. Example, . . . . .	62
35.	<i>Backward</i> interpolation by STIRLING'S Formula. Example, . . . .	65
36.	Further example in the use of STIRLING'S Formula, . . . . .	65
37.	The <i>algebraic mean</i> . Practical precepts, . . . . .	66
38.	Derivation of BESSEL'S Formula. Numerical application, . . . .	67
39.	Second example of interpolation by BESSEL'S Formula, . . . . .	68
40.	<i>Backward</i> interpolation by BESSEL'S Formula. Example, . . . . .	69
41.	Property of BESSEL'S Coefficients, . . . . .	69
42.	Comparison of the relative advantages of NEWTON'S, STIRLING'S, and BESSEL'S Formulae, . . . . .	71
43.	Simple interpolation. Magnitude of error arising from neglect of second differences, . . . . .	72
44.	Interpolation by means of a <i>corrected first difference</i> . Example, . .	73
45.	<i>Backward</i> interpolation by means of a corrected first difference. Examples, . . . . .	74
46, 47.	Correction of erroneous tabular functions by direct interpolation. Example, . . . . .	76
48.	Systematic interpolation of series. Reduction of a given tabular in- terval, . . . . .	78
49.	Interpolation <i>to halves</i> . Practical rule, . . . . .	80
50.	Precepts for systematic interpolation <i>to halves</i> . Schedule showing arrangement of quantities. Numerical example, . . . . .	81
51.	Derivation of general formulae for reducing the tabular interval from $\omega$ to $m\omega$ , $m$ being the reciprocal of a positive <i>odd</i> integer, . . .	83

# CONTENTS.

vii

Section.		Page.
52.	Systematic interpolation <i>to thirds</i> . Example, . . . . .	88
53.	Systematic interpolation <i>to fifths</i> . Example, . . . . .	89
54.	On the best order of performing successive interpolations to halves, thirds, etc., . . . . .	91
55.	Interpolation, with a constant interval $n$ , of an entire series of functions. Example, . . . . .	91
	Examples, . . . . .	94

## CHAPTER III.

### DERIVATIVES OF TABULAR FUNCTIONS.

56.	Concerning the close relation between differences and differential coefficients, . . . . .	97
57.	Practical applications of formulae resulting from this relation. Importance of tabular derivatives in Astronomy, . . . . .	97
58.	Derivation of the required formulae in general terms, . . . . .	98
59.	Formulae for computing derivatives at or near the <i>beginning</i> of a series. Examples, . . . . .	101
60.	Formulae applicable at or near the <i>end</i> of a series. Examples, . . . . .	105
61.	Derivatives from STIRLING'S Formula. Rule for computing $F'(t)$ . Examples, . . . . .	109
62.	Derivatives from BESSEL'S Formula. Simple expression for $F'(t + \frac{1}{2}\omega)$ . Applications and examples, . . . . .	115
63.	Interpolation by means of tabular first derivatives. Example, . . . . .	121
64.	Application of preceding method when second differences are nearly constant. Practical rule for this case. Examples, . . . . .	124
65.	Regarding a choice of formulae in any given case, . . . . .	127
	Examples, . . . . .	128

## CHAPTER IV.

### OF MECHANICAL QUADRATURE.

66.	Statement of the problem. Important applications of the method, . . . . .	130
67.	Derivation of formulae for single integration from NEWTON'S Formula. The auxiliary series $'F$ . Schedule of functions and differences, . . . . .	131
68.	Numerical applications illustrating two of the foregoing formulae, . . . . .	137
69.	Precepts for computing a definite integral when either or both limits are other than <i>tabular</i> values of the argument $T$ . Necessity of interpolation in this case, . . . . .	138

Section.		Page.
70, 71.	Transformation and extension of the fundamental relations of §67, such that integrals whose limits are <i>non-tabular</i> values of $T$ are expressed directly in terms of <i>interpolated</i> values of $'F, F, \Delta', \Delta'', \Delta'''$ , etc. Formulae and examples, . . . . .	140
72.	Formulae for single integration as derived from STIRLING'S Formula. Schedule of functions and differences. Examples, . . . . .	146
73.	Generalization of preceding formulae to include integrals of <i>any</i> limits. Example, . . . . .	151
74.	Formulae for single integration from BESSEL'S Formula. Extension to <i>any</i> limits. Examples, . . . . .	153
75.	Double integration. The conditions involved, . . . . .	160
76.	Derivation of formulae for double integration from NEWTON'S Formula. Introduction of the series $''F$ . Schedule of functions and differences. General formulae and relations, . . . . .	160
77.	Value of the <i>first</i> integral at the lower limit. Introduction and definition of the quantity $H_0$ . Collection of formulae for double integration covering all possible cases. Examples, . . . . .	166
78.	Derivation of formulae for double integration from STIRLING'S and BESSEL'S Formulae. Schedule referring to same. Precepts and examples, . . . . .	173
79.	Change in value of the double integral $Y$ , due to an arbitrary change in the constant $H$ , . . . . .	188
	Examples, . . . . .	189

## CHAPTER V.

## MISCELLANEOUS PROBLEMS AND APPLICATIONS.

80.	Introductory statement, . . . . .	191
81.	Problem I.—To find the sum of the $k$ th powers of the first $r$ integers. Application to $S \equiv 1^4 + 2^4 + 3^4 + \dots + r^4$ , . . . . .	191
82, 83.	Problem II.—Given the series $F_{-2}, F_{-1}, F_0, F_1, F_2$ , etc., and an assigned value of $F_n$ ; to find the corresponding interval $n$ . Two solutions. Examples, . . . . .	192
84.	Problem III.—To solve any numerical equation containing but one unknown quantity. Example, . . . . .	195
85.	Problem IV.—To find the value of the argument corresponding to a maximum or minimum function. Example, . . . . .	196
86.	Problem V.—Given a series of values, $F_{-2}, F_{-1}, F_0, F_1, F_2$ , etc., of some function $F(T)$ analytically unknown; to find an approximate algebraic expression for $F(T)$ . Examples, . . . . .	198
87.	Geometrical problem, . . . . .	200
88.	Concluding remarks, . . . . .	202
	Examples, . . . . .	203

## APPENDIX.

## ON THE SYMBOLIC METHOD OF DEVELOPMENT.

Section.		Page.
89.	Introductory remarks, . . . . .	205
90.	Definition and operation of the symbols $\Delta$ , $\Delta^2$ , $\Delta^3$ , etc., . . . . .	205
91.	Definition and operation of $D$ , $D^2$ , $D^3$ , etc., . . . . .	206
92, 93.	Proof that the foregoing symbols of operation obey, in general, the fundamental laws of algebraic combination, . . . . .	206
94.	Consideration of negative powers of $\Delta$ and $D$ , . . . . .	208
95.	Remark concerning results established in the preceding sections, . . . . .	209
96.	Demonstration of Theorem III, . . . . .	209
97.	Fundamental relation between $\Delta$ and $D$ , . . . . .	210
98.	Expression of $\Delta$ , $\Delta^2$ , $\Delta^3$ , etc., in terms of ascending powers of $D$ . Demonstration of Theorem V, . . . . .	210
99.	Expression of $D$ , $D^2$ , $D^3$ , etc., in terms of ascending powers of $\Delta$ , . . . . .	211
100.	Reduction of the tabular interval $\omega$ . Expression of $\partial$ , $\partial^2$ , $\partial^3$ , etc., in terms of ascending powers of $\Delta$ , . . . . .	211
101.	Effect of the operator $1 + \Delta$ . NEWTON'S Formula of interpolation, . . . . .	211
102.	Definition of the symbol of operation $\nabla$ . Its relation to $\Delta$ and $D$ , . . . . .	212
103.	Derivation of NEWTON'S Formula for <i>backward</i> interpolation, . . . . .	213
104.	Expression of any difference in terms of the given tabular functions, . . . . .	213
105.	Derivation of the fundamental relations of mechanical quadrature. Single integration, . . . . .	214
106.	The fundamental formulae of double integration, . . . . .	214

## TABLES.

TABLE I.	NEWTON'S coefficients of interpolation, . . . . .	218
TABLE II.	STIRLING'S coefficients of interpolation, . . . . .	220
TABLE III.	BESSEL'S coefficients of interpolation, . . . . .	222
TABLE IV.	NEWTON'S coefficients for computing $F'(T)$ , . . . . .	224
TABLE V.	STIRLING'S coefficients for computing $F'(T)$ , . . . . .	226
TABLE VI.	BESSEL'S coefficients for computing $F'(T)$ , . . . . .	228
TABLE VII.	Giving $y$ : For finding $n$ when $F_n$ is given, . . . . .	230
TABLE VIII.	Coefficients for interpolating by means of tabular first derivatives, . . . . .	232
BIBLIOGRAPHY,	. . . . .	233



## CHAPTER I.

### OF DIFFERENCES.

1. In many applications of the exact sciences, and of Astronomy in particular, it is often necessary to tabulate a series of numerical values of some quantity or function, corresponding to certain assumed values of the element or argument upon which the functional values depend.

In the more purely mathematical tables, the function is analytically known; the argument is then the independent variable of the given expression. The common tables of logarithms, trigonometrical functions, squares, cubes, and reciprocals, are examples of tabular functions of this class.

A second and larger class includes those functions which are not related analytically to the argument, but which are either determined directly by experiment, or based wholly or partly upon observation. The final results are usually obtained from the fundamental observations by suitable mathematical transformations or reductions, which frequently include the process of adjustment known as the method of least-squares. Empirical values are also occasionally introduced in the development of functions of this class, to supply some theoretical deficiency.

In the great majority of such cases, the *time* is the argument of the tabulated function. This is particularly the case in astronomical tables. Thus the *Nautical Almanac* gives the right-ascensions and declinations of the sun and the planets for every Greenwich mean noon; in the case of the moon, these coördinates are given for every hour, because of the rapid motion of our satellite. The moon's horizontal parallax is tabulated for every twelve hours; the sun's for every ten days.

In like manner, the readings of the barometer and thermometer

are recorded for certain hours of the day, and therefore may be regarded as functions of the time. The velocity of the wind, the height of tide-water, the correction and rate of a clock, are further instances of a large number of physical quantities which are tabulated as functions of the time.

As examples of tabular functions of the physical or observational kind, whose arguments are elements other than the time, we may mention :

(a) The force of gravity (determined by pendulum experiments), as a function of the latitude ;

(b) The atmospheric pressure (determined by the barometer), as a function of the altitude ;

(c) The angle of refraction in a particular substance, as a function of the angle of incidence.

Although differing thus fundamentally in the character of their respective functions, all mathematical tables are alike in giving the numerical values of the functions for certain assumed values of the argument, so chosen that intermediate values of the function may readily be derived by the process of *interpolation*. For this purpose it is convenient, though not essential, to have the assumed argument values proceed according to some law ; and since as a rule the greatest simplicity is attained where the argument varies uniformly, it is nearly always so taken. The *interval* of the argument is decided in general by the rapidity with which the given function varies.

We shall assume throughout these pages that the given values of the argument are equidistant.

The present chapter will be devoted to the subject of *differences*, as defined below. The student should become thoroughly and practically familiar with this fundamental portion of the work before entering upon the chapters that follow.

2. *Definitions and Notation*.—If we have given a series of quantities proceeding according to any law, and take the difference of every two consecutive terms, we obtain a series of values called the *first order of differences*, or briefly, *first differences*.

If we difference the first differences in the same manner, we form a new series called *second differences*. The process may be continued, if necessary, so long as any differences remain.

We shall apply this process of differencing to the tabular values of functions given for equidistant values of the argument.

Let  $T$  designate the argument;  $\omega$ , its interval;  $F(T)$ , or simply  $F$ , the function;  $t, t + \omega, t + 2\omega, t + 3\omega, \dots$ , the given values of  $T$ ;  $F_0, F_1, F_2, F_3, \dots$ , the corresponding values of  $F(T)$ ;  $\Delta', \Delta'', \Delta''', \Delta^{iv}, \dots$ , the successive orders of differences. The arrangement is then shown in the following schedule:

Argument	Function	1st Diff. 2d Diff.		3d Diff.	4th Diff.	5th Diff.	6th Diff.
$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$
$t$	$F_0$						
$t + \omega$	$F_1$	$a_0$	$b_0$				
$t + 2\omega$	$F_2$	$a_1$	$b_1$	$c_0$	$d_0$		
$t + 3\omega$	$F_3$	$a_2$	$b_2$	$c_1$	$d_1$	$e_0$	
$t + 4\omega$	$F_4$	$a_3$	$b_3$	$c_2$	$d_2$	$e_1$	$f_0$
$t + 5\omega$	$F_5$	$a_4$	$b_4$	$c_3$			
$t + 6\omega$	$F_6$	$a_5$					

where  $a_0 = F_1 - F_0, a_1 = F_2 - F_1, \dots; b_0 = a_1 - a_0, b_1 = a_2 - a_1, \dots; c_0 = b_1 - b_0, c_1 = b_2 - b_1, \dots$ ; and so on.

We shall also find it convenient to represent  $a_0, a_1, a_2, \dots$  by  $\Delta'_0, \Delta'_1, \Delta'_2, \dots$ , respectively;  $b_0, b_1, b_2, \dots$  by  $\Delta''_0, \Delta''_1, \Delta''_2, \dots$ , etc., Thus, generally,  $\Delta_s^{(n)}$  denotes the  $(s+1)^{\text{th}}$  value in the column of  $n^{\text{th}}$  differences.

As an example, we tabulate and difference several successive values of  $F(T) \equiv T^4 - 10T^2 - 20$ , thus:

<i>T</i>	<i>F(T)</i>	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
0	— 20	— 9				
1	— 29	— 15	— 6	+ 36		
2	— 44	+ 15	+ 30	+ 60	+24	
3	— 29	+105	+ 90	+ 84	+24	0
4	+ 76	+279	+174	+108	+24	0
5	+355	+561	+282			
6	+916					

The differences are in all cases formed by subtracting (algebraically) downwards, as in the above examples. It will be noted that the even differences ( $\Delta'', \Delta^{iv}, \dots$ ) always fall on the same lines with the argument and function, while the odd differences ( $\Delta', \Delta''', \Delta^v, \dots$ ) lie between the lines.

3. *Method of Checking the Numerical Accuracy of the Differences.*— If, in the numerical example of the last section, we take the algebraic sum of the six given values of  $\Delta'$ , we find

$$-9 - 15 + 15 + 105 + 279 + 561 = +936$$

Subtracting the first value of  $F(T)$  from the last, we have

$$+916 - (-20) = +936$$

which agrees with the first result.

Again, in like manner, we find

$$\Delta_0''' + \Delta_1''' + \Delta_2''' = +36 + 60 + 84 = +180 = +174 - (-6) = \Delta_3'' - \Delta_0''$$

These relations may be expressed generally as follows :

THEOREM I.— *The algebraic sum of any  $s$  consecutive values of  $\Delta^{(n)}$ , is equal to the last, minus the first, of the  $s+1$  consecutive  $\Delta^{(n-1)}$  terms used in forming the  $s$  values of  $\Delta^{(n)}$ .*

To prove this proposition, let the differences be as below :

$$\begin{array}{lcl} \Delta^{(n-1)} : & h_1 & h_2 \ h_3 \ . \ . \ . \ . \ . \ h_{s-1} \ h_s \ h_{s+1} \\ \Delta^{(n)} : & k_1 & k_2 \ k_3 \ . \ . \ . \ . \ . \ k_{s-1} \ k_s \end{array}$$

Then, from the definition of differences, we have

$k_1 = h_2 - h_1, \quad k_2 = h_3 - h_2, \quad . . . . ., \quad k_{s-1} = h_s - h_{s-1}, \quad k_s = h_{s+1} - h_s$

Hence, by addition, we find

$k_1 + k_2 + k_3 + . . . . . + k_{s-1} + k_s = h_{s+1} - h_1$

which is the algebraic statement of Theorem I. This theorem may obviously be applied as an independent check upon the numerical accuracy of the differencing.

4. THEOREM II.—*If the differences of  $N$  values of  $F(T)$  are taken,  $N-n$  values of  $\Delta^{(n)}$  are derived; it being assumed that  $N > n$ .*

For,  $N$  functions evidently yield  $N-1$  values of  $\Delta$ ,  $N-2$  values of  $\Delta''$ ,  $N-3$  values of  $\Delta'''$ , etc.; hence  $N$  values of  $F(T)$  yield  $N-n$  values of  $\Delta^{(n)}$ .

5. *Inversion of a Series of Functions.*—It is sometimes necessary or convenient to invert a given column of functions, thus bringing the last value into the position of the first, the next to the last into the position of the second, etc. For example, let us invert the series given in §2, and observe the effect of this inversion upon the differences. Thus we find :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
6	+ 916	— 561				
5	+ 355	— 279	+ 282			
4	+ 76	— 105	+ 174	— 108	+ 24	
3	— 29	— 15	+ 90	— 84	+ 24	0
2	— 44	+ 15	+ 30	— 60	+ 24	0
1	— 29	+ 9	— 6	— 36		
0	— 20					

Comparing this table with the original, we first observe that each column of differences is inverted, like the column of functions itself. Further, having regard to signs, we see that the first and third differences (the *odd* orders) have changed signs throughout; while  $\Delta''$  and  $\Delta^{iv}$  (the *even* orders) remain unaltered in sign.

To prove that such an effect is true generally, we consider the two series below, the second series being an inversion of the first :

$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$F_0$				
$F_1$	$a_0$			
$F_2$	$a_1$	$b_0$		
$F_3$	$a_2$	$b_1$	$c_0$	$d_0$
$F_4$	$a_3$	$b_2$	$c_1$	$d_1$
$F_5$	$a_4$	$b_3$	$c_2$	

$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$F_5$				
$F_4$	$\alpha_0$			
$F_3$	$\alpha_1$	$\beta_0$		
$F_2$	$\alpha_2$	$\beta_1$	$\gamma_0$	$\delta_0$
$F_1$	$\alpha_3$	$\beta_2$	$\gamma_1$	$\delta_1$
$F_0$	$\alpha_4$	$\beta_3$	$\gamma_2$	

Comparing the first differences, we find

$$\begin{aligned} \alpha_0 &= F_4 - F_5 = -(F_5 - F_4) = -a_4 \\ \alpha_1 &= F_3 - F_4 = -(F_4 - F_3) = -a_3 \\ \alpha_2 &= F_2 - F_3 = -(F_3 - F_2) = -a_2 \\ &\dots \end{aligned}$$

Hence, for the second differences, we obtain

$$\begin{aligned} \beta_0 &= \alpha_1 - \alpha_0 = -a_3 - (-a_4) = a_4 - a_3 = b_3 \\ \beta_1 &= \alpha_2 - \alpha_1 = -a_2 - (-a_3) = a_3 - a_2 = b_2 \\ &\dots \end{aligned}$$

Thus, the inversion of the functions inverts  $\Delta'$ , and changes its signs throughout ; whereas  $\Delta''$  is inverted, but does not change in sign, Further, since  $\Delta'''$  and  $\Delta^{iv}$  have the same relation to  $\Delta''$ , that  $\Delta'$  and  $\Delta''$  have to  $F(T)$ , it is manifest that  $\Delta'''$  inverts and changes signs, while  $\Delta^{iv}$  inverts with signs unaltered. Extending this reasoning, we have the following proposition :

THEOREM III.—*Inverting a series of functions inverts each column of differences and changes the signs of the odd orders ( $\Delta'$ ,  $\Delta'''$ ,  $\Delta^v$ , . . . .), while the signs of the even orders ( $\Delta''$ ,  $\Delta^{iv}$ , . . . .) remain unchanged.*

In practice it is seldom necessary to re-tabulate the function in the inverted order, since we may readily conceive the inversion to be made, merely allowing for the changes of sign in  $\Delta'$ ,  $\Delta'''$ ,  $\Delta^v$ , etc.

6. THEOREM IV.—*The  $n^{th}$  differences of the sums of two series of functions are equal to the sums of the corresponding  $n^{th}$  differences of the two component series.*

To prove generally, let  $F_0, F_1, F_2, \dots$ , and  $f_0, f_1, f_2, \dots$ , denote the two series of functions; then the sums of the two series will be  $F_0+f_0, F_1+f_1, F_2+f_2, \dots$ . Also, let us designate the first differences of these three series by  $\Delta'$ ,  $\delta'$ , and  $D'$ , respectively; their values are hence as follows:

$\Delta'$

$$\begin{array}{c|c} F_0 & F_1-F_0 \\ F_1 & F_2-F_1 \\ F_2 & \vdots \\ \vdots & \vdots \end{array}$$

$\delta'$

$$\begin{array}{c|c} f_0 & f_1-f_0 \\ f_1 & f_2-f_1 \\ f_2 & \vdots \\ \vdots & \vdots \end{array}$$

$D'$

$$\begin{array}{c|c} F_0+f_0 & (F_1+f_1)-(F_0+f_0) \\ F_1+f_1 & (F_2+f_2)-(F_1+f_1) \\ F_2+f_2 & \vdots \\ \vdots & \vdots \end{array}$$

We therefore have

$$D_0' = (F_1+f_1) - (F_0+f_0) = F_1+f_1 - F_0-f_0 = (F_1-F_0) + (f_1-f_0) = \Delta_0' + \delta_0'$$

$$D_1' = (F_2+f_2) - (F_1+f_1) = F_2+f_2 - F_1-f_1 = (F_2-F_1) + (f_2-f_1) = \Delta_1' + \delta_1'$$

$$\dots\dots\dots$$

These relations prove the theorem directly for  $n = 1$ ; but since the second differences are formed from the first differences in the same manner that the latter are derived from the given functions, the theorem is also true for  $n = 2$ . Similarly with the following differences, each order being the first difference of the order just preceding. Hence the theorem is true generally.

As an example we write:

$F$	$\Delta'$	$\Delta''$	$\Delta'''$	$f$	$\delta'$	$\delta''$	$\delta'''$	$F+f$	$D'$	$D''$	$D'''$
- 5	+ 1			+14				+ 9			
- 4	+13	+12		+16	+2			+ 12	+ 3		
+ 9	+31	+18	+6	+19	+3	+1		+ 28	+16	+13	
+40	+55	+24	+6	+19	0	-3	-4	+ 59	+31	+15	+2
+95				+13	-6	-6	-3	+108	+49	+18	+3

It will be observed that the values of  $D'$ ,  $D''$  and  $D'''$  are in accord with the theorem.

7. *Irregularities in the Differences.*—In the numerical example of §2, the values of  $\Delta^n$  are all zero. In such a case, we say that the differences are perfectly smooth or *regular*. In practice, however, the

differences frequently exhibit a small degree of irregularity, owing to the omission of decimals in the approximate values of the functions employed. As an example, we take the following values of  $T^4$ , true to the nearest unit of the second decimal :

$T$	$F(T) \equiv T^4$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
2.0	16.00				
2.1	19.45	+3.45	+0.53		
2.2	23.43	3.98	.57	+0.04	
2.3	27.98	4.55	.65	.08	+0.04
2.4	33.18	5.20	.68	.03	— .05
2.5	39.06	5.88	.76	.08	+ .05
2.6	45.70	6.64	.80	.04	— .04
2.7	53.14	7.44	+0.89	+0.09	+0.05
2.8	61.47	+8.33			

That the irregularity here manifest in the outer differences is due to the fact that the tabular values are only approximate (not the true mathematical values of the function), may easily be shown by Theorem IV, thus : let

$\bar{F}$  denote the true value of the function ;

$F$ , its approximate value as above ;

$f = F - \bar{F}$ , the difference of these values.

Then, since  $F$  is given to the nearest unit of the second place,  $f$  may have any value from  $-0.5$  to  $+0.5$ , in terms of the same unit. Moreover, the values of  $f$  do not follow any law of progression, but proceed at random, with arbitrary changes of sign. Hence, the *differences* of  $f$  will be irregular. The differences of  $\bar{F}$  must proceed regularly, however, since  $\bar{F}$  is the true mathematical value of a continuous function. Now, since  $F = \bar{F} + f$ , it follows from Theorem IV that the differences of  $F$  must equal the sums of the corresponding differences of  $\bar{F}$  and  $f$ ; therefore, *the differences of  $F$  must contain just such irregularities as are inevitable in the differences of  $f$ .*

To illustrate this principle, we tabulate below the values of  $\bar{F}$ , along with the given series,  $F$ ; whence  $f$  follows, in units of the second decimal, and also the differences of  $f$  to the fourth order :

$T$	$\bar{F}(T)$	$F(T)$	$f = F - \bar{F}$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
2.0	16.00,00	16.00	0.00				
2.1	19.44,81	19.45	+0.19	+0.19	+0.06		
2.2	23.42,56	23.43	+0.44	+0.25	-1.10	-1.16	+3.76
2.3	27.98,41	27.98	-0.41	-0.85	+1.50	+2.60	-5.24
2.4	33.17,76	33.18	+0.24	+0.65	-1.14	-2.64	+4.76
2.5	39.06,25	39.06	-0.25	-0.49	+0.98	+2.12	-4.24
2.6	45.69,76	45.70	+0.24	+0.49	-1.14	-2.12	+4.76
2.7	53.14,41	53.14	-0.41	-0.65	+1.50	+2.64	
2.8	61.46,56	61.47	+0.44	+0.85			

We now bring together, from the above tables, the fourth differences of  $F$  and  $f$ , denoting these quantities by  $(\Delta^{iv})F$  and  $(\Delta^{iv})f$ , respectively. The fourth differences of  $\bar{F}$  then follow, since we have shown that  $(\Delta^{iv})F = (\Delta^{iv})\bar{F} + (\Delta^{iv})f$ ; thus we form the table below :

$(\Delta^{iv})F$	$(\Delta^{iv})f$	$(\Delta^{iv})\bar{F}$
+0.04	+0.03,76	+0.0024
-0.05	-0.05,24	+0.0024
+0.05	+0.04,76	+0.0024
-0.04	-0.04,24	+0.0024
+0.05	+0.04,76	+0.0024

It will be observed that the fourth differences of  $\bar{F}(T)$  are absolutely uniform,—that is, the irregularities in  $(\Delta^{iv})F$  and  $(\Delta^{iv})f$  exactly correspond, or balance. The slight irregularity in the outer differences of the series  $F(T)$  is therefore due entirely to the omission of decimals, since it wholly disappears when we employ the true mathematical values,  $\bar{F}(T)$ .

As a valuable exercise, the student should now difference the function  $\bar{F}$  directly, and find the fourth differences exactly as above deduced.

8. *Detection of Accidental Errors.*—We have just seen how a slight deviation from the true value of a tabular function will manifest itself by means of irregularities in the differences. If, then, some one value of a series is in error by an appreciable quantity, an inspection of the differences will indicate definitely the location and magnitude of the error sought.

To investigate the principle that underlies the method, let  
 $F_0, F_1, F_2, F_3, F_4, F_5, \dots$   
denote the *correct* values of any function  $F(T)$  (tabulated for equi-  
distant values of  $T$ ), and let the differences be as shown in the  
schedule below :

$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$F_0$	$a_0$				
$F_1$	$a_1$	$b_0$			
$F_2$	$a_2$	$b_1$	$c_0$	$d_0$	$e_0$
$F_3$	$a_3$	$b_2$	$c_1$	$d_1$	$e_1$
$F_4$	$a_4$	$b_3$	$c_2$	$d_2$	$e_2$
$F_5$	$a_5$	$b_4$	$c_3$	$d_3$	$e_3$
$F_6$	$a_6$	$b_5$	$c_4$	$d_4$	$e_4$
$F_7$	$a_7$	$b_6$	$c_5$	$d_5$	$e_5$
$F_8$	$a_8$	$b_7$	$c_6$	$d_6$	$e_6$
$F_9$	$a_9$	$b_8$	$c_7$	$d_7$	$e_7$
$F_{10}$	$a_{10}$	$b_9$	$c_8$	$d_8$	
$F_{11}$	$a_{11}$	$b_{10}$	$c_9$		
$F_{12}$					

Let us now assume that some one function, say  $F_6$ , is in error  
by the quantity  $\epsilon$ , so that  $F_6 + \epsilon$  is tabulated in place of the true  
value  $F_6$ ; the differences of the incorrect series will therefore be found  
as follows :

$F(T) + \epsilon$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$F_0$	$a_0$				
$F_1$	$a_1$	$b_0$			
$F_2$	$a_2$	$b_1$	$c_0$	$d_0$	$e_0$
$F_3$	$a_3$	$b_2$	$c_1$	$d_1$	$e_1 + \epsilon$
$F_4$	$a_4$	$b_3$	$c_2 + \epsilon$	$d_2 - 4\epsilon$	$e_2 - 5\epsilon$
$F_5$	$a_5 + \epsilon$	$b_4 + \epsilon$	$c_3 - 3\epsilon$	$d_3 + 6\epsilon$	$e_3 + 10\epsilon$
$F_6 + \epsilon$	$a_6 - \epsilon$	$b_5 - 2\epsilon$	$c_4 + 3\epsilon$	$d_4 - 4\epsilon$	$e_4 - 10\epsilon$
$F_7$	$a_7$	$b_6 + \epsilon$	$c_5 - \epsilon$	$d_5 + \epsilon$	$e_5 + 5\epsilon$
$F_8$	$a_8$	$b_7$	$c_6$	$d_6$	$e_6 - \epsilon$
$F_9$	$a_9$	$b_8$	$c_7$	$d_7$	$e_7$
$F_{10}$	$a_{10}$	$b_9$	$c_8$	$d_8$	
$F_{11}$	$a_{11}$	$b_{10}$	$c_9$		
$F_{12}$					

Now, because the differences of the correct table contain no  
irregularities, we see that the differences of the incorrect table consist  
of series of regular values, to which are alternately added and sub-  
tracted the terms in  $\epsilon$ , shown in the above schedule. The law of  
progression and increase in the coefficients of  $\epsilon$ , along the successive

orders of differences, is easily seen to be that of the binomial coefficients, with alternate signs. Hence, in practice, we have only to carry the differencing to that order at which the differences of the correct functions would vanish, or sensibly so; the location and magnitude of the error will then be clearly shown by a succession of + and — terms, following the binomial law.

Thus, if the values of  $\Delta^v$  vanish in the correct table above, the fifth differences of the incorrect series will be 0,  $+\epsilon$ ,  $-5\epsilon$ ,  $+10\epsilon$ ,  $-10\epsilon$ ,  $+5\epsilon$ ,  $-\epsilon$ , 0; the initial value,  $+\epsilon$ , is therefore the error sought, both as to magnitude and sign. The required function is found by tracing backwards and downwards along the line of heavy type from  $e_1+\epsilon$  to  $F_6+\epsilon$ , which is the incorrect function; and since the *correction* is the negative of the error, we have  $(F_6+\epsilon)-\epsilon$ , or  $F_6$ , for the true value of the function in question.

9. We shall now consider several examples, in order that the process may be fully understood.

EXAMPLE I. — Find the error in the following table of  $F(T) \equiv T^3$ :

$T$	$F(T) \equiv T^3$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$c$	$\Delta^{iv}+c$
1	1						
2	8	+ 7	+12				
3	27	19	18	+ 6			
4	64	37	24	+ 6	0		0
5	125	61	24	— 4	—10	+10	0
6	<b>206</b>	<b>81</b>	<b>20</b>	+36	+40	—40	0
7	343	<b>137</b>	56	—24	—60	+60	0
8	512	169	<b>32</b>	—24	+40	—40	0
9	729	217	48	+16	+40	+10	0
10	1000	+271	+54	+ 6	—10		

The differencing is continued until we find a complete alternation of signs, as in  $\Delta^{iv}$ . Now the binomial coefficients of the fourth order are 1, 4, 6, 4, 1; it is also seen that the values of  $\Delta^{iv}$  are just these numbers multiplied by 10. Hence, an error of 10 units exists somewhere in the function  $F$ ; its location is easily determined by tracing backwards and downwards along the line of  $-10, -4, +20, +81$ , to the number 206, which is the quantity sought. The required function is also found by tracing backwards and upwards along the line of  $-10, +16, +32, +137$ , to 206; in practice, both lines should be followed, to guard against mistake.

Finally, the number 206 is *too small* by 10 units, since the *sign* of the error is shown by the leading or initial value of the binomial series in  $\Delta^v$ , namely,  $-10$ . A correction of  $+10$  is therefore to be applied to the incorrect function, giving 216 for its true value.

In the column  $c$ , following  $\Delta^v$  in the above table, are given the corrections to  $\Delta^v$ , due to the correction of  $+10$  to the function. The column  $\Delta^v+c$  therefore gives the 4th differences of the true or corrected series. It is always well to re-difference the series after a correction has been applied, to check the accuracy of the work.

EXAMPLE II.—Find the error in the following table of logarithms:

$T$	$\log T$	$\Delta'$	$\Delta''$	$\Delta'''$	$c$	$\Delta''' + c$
45	1.6532					
50	1.6990	+458	—44			
55	1.7404	414	<b>31</b>	<b>+13</b>	— 5	+8
60	<b>1.7787</b>	<b>383</b>	41	—10	+15	5
65	1.8129	<b>342</b>	<b>20</b>	+21	—15	6
70	1.8451	322	<b>22</b>	— 2	+ 5	3
75	1.8751	300	+20	+ 2		+2
80	1.9031	+280				

The third differences are here sufficient to point out the error ; the correction given under  $c$  appears to improve  $\Delta'''$  in the best manner, thus indicating that  $\log 60$  should be 1.7782 instead of 1.7787. It will be observed that a correction of  $-6$  is nearly as efficient as  $-5$  in the above case, and that  $-5.5$  is better than either ; this is because the value of  $\log 60$  to five places is 1.77815.

EXAMPLE III.—Correct the error in the following ephemeris of the moon's latitude :

Date 1898	Moon's Lat. $\beta$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$c$	$\Delta^v+c$
	$^{\circ} \quad ' \quad ''$	$' \quad ''$	$' \quad ''$	$''$	$''$	$''$	$''$	$''$
May 8.5	—1 59 54.2	+37 10.0						
9.0	1 22 44.2	38 17.2	+1 7.2					
9.5	0 44 27.0	38 41.7	+0 24.5	—42.7				
10.0	—0 5 45.3	38 25.2	—0 16.5	41.0	+ 1.7	+ <b>14.1</b>	— 12.8	+1.3
10.5	+0 32 39.9	<b>37 43.5</b>	<b>0 41.7</b>	<b>25.2</b>	—47.6	— 63.4	+ 64.0	0.6
11.0	<b>1 10 23.4</b>	<b>35 49.0</b>	1 54.5	—72.8	+80.6	+128.2	—128.0	0.2
11.5	1 46 12.4	<b>34 2.3</b>	<b>1 46.7</b>	+ 7.8	—46.9	—127.5	+128.0	+0.5
12.0	2 20 14.7	+31 36.5	—2 25.8	<b>—39.1</b>				
12.5	+2 51 51.2				••••	•••••	•••	•••

In this example the error is readily indicated in  $\Delta^v$ , for which order the binomial coefficients are 1, 5, 10, 10, 5, 1. Although but

four values of  $\Delta'$  are available, these are here sufficient. A slight inspection shows that a correction of  $-13''.0$ , as applied to the latitude for May 11.0, will very nearly serve the purpose;  $-13''.0$  being a trifle too great numerically, we soon find by trial that  $-12''.8$  produces the best result. Hence, the moon's latitude for May 11.0 should read,  $+1^\circ 10' 10''.6$ .

10. *Correction of Errors when More than One Function is Affected.*—Thus far we have considered examples of an error in one function only. When two or more consecutive or neighboring values are in error, the problem of correction becomes more complicated and difficult. It may even become indeterminate in some cases, since only *accidental* errors can be detected by the differences. Several successive functions, and possibly all, may contain systematic errors which do not affect the regularity of the differences.

In general, the correction of a group of errors by differences may be considered practicable only when the law of the function is not obscured or altered by the presence of those errors. More definitely, the method may be regarded as available in the case of two or perhaps three neighboring functions, provided the errors be accidental in character, and of sufficient magnitude to produce a distinct and definitive irregularity in the differences.

EXAMPLE I.—Correct the errors in the following tabulation of  $F(T) \equiv 2T^3 - 25T - 40$ :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$c_1$	$\Delta''' + c_1$	$c_2$	$\Delta''' + c_1 + c_2$
-4	- 68	+ 49						
3	19	+ 13	-36	+12		+12		+12
2	6	- 11	24	12		12		12
-1	17	23	-12	12		12		12
0	40	23	0	12		12		12
+1	63	23	+ 7	7	+ 5	12		12
2	79	- 16	27	27	-15	12		12
3	61	+ 18	34	+ 5	+15	+20	- 8	12
4	- 4	57	39	- 7	- 5	-12	+24	12
5	+ 85	89	32	+36		+36	-24	12
6	242	157	68	4		4	+ 8	12
7	471	229	72	+12		+12		+12
+8	+784	+313	+84					

We carry the differences to the third order, and note that the first three values of  $\Delta'''$  are constant, and equal to  $+12$ ; hence, in

column  $c_1$ , we place the correction of  $+5$ . This gives a corrected series for  $\Delta'''$ , shown under  $\Delta''' + c_1$ . The latter column clearly indicates a correction of  $-8$ , as applied in  $c_2$ ; this gives a final corrected column of third differences, with the constant value of  $+12$ . Hence, the value  $F(T)$  for  $T = +2$ , should read  $-74$  instead of  $-79$ ; for  $T = +4$ , we should have  $-12$  instead of  $-4$ .

EXAMPLE II.—Correct the errors which occur in the following ephemeris of the sun's declination :

Date 1898	Sun's Decl. $\delta$	$\Delta'$	$\Delta''$	$\Delta'''$	$c_1 \text{ \& } c_2$	$\Delta''' + c_1 + c_2$	$c_3$	$\Delta''' + c_1 + c_2 + c_3$
	$^{\circ} \quad ' \quad ''$	$\quad \quad ' \quad ''$	$\quad \quad ''$	$\quad \quad ''$		$\quad \quad ''$		$\quad \quad ''$
Jan. 28	$-18 \ 6 \ 34.7$	$+32 \ 30.7$						
30	$17 \ 34 \ 4.0$	$33 \ 45.0$	$+74.3$					
Feb. 1	$17 \ 0 \ 19.0$	$34 \ 56.1$	$71.1$	$-3.2$		$-3.2$		$-3.2$
3	$16 \ 25 \ 22.9$	$36 \ 4.1$	$68.0$	$-3.1$		$-3.1$		$3.1$
5	$15 \ 49 \ 18.8$	$37 \ 12.2$	$68.1$	$+0.1$	$-3.2$	$-3.1$		$3.1$
7	$15 \ 12 \ 6.6$	$38 \ 12.6$	$60.4$	$-7.7$	$+9.6$	$+1.9$	$-5.1$	$3.2$
9	$14 \ 33 \ 54.0$	$39 \ 1.2$	$48.6$	$-11.8$	$-9.6 + 3.0$	$-18.4$	$+15.3$	$3.1$
11	$13 \ 54 \ 52.8$	$40 \ 7.8$	$66.6$	$+18.0$	$+3.2 - 9.0$	$+12.2$	$-15.3$	$3.1$
13	$13 \ 14 \ 45.0$	$40 \ 56.9$	$49.1$	$-17.5$	$+9.0$	$-8.5$	$+5.1$	$3.4$
15	$12 \ 33 \ 48.1$	$41 \ 45.7$	$48.8$	$-0.3$	$-3.0$	$-3.3$		$3.3$
17	$11 \ 52 \ 2.4$	$+42 \ 31.0$	$+45.3$	$-3.5$		$-3.5$		$-3.5$
19	$-11 \ 9 \ 31.4$							

In this case, the first, second, and last values of  $\Delta'''$  are  $-3.2$ ,  $-3.1$  and  $-3.5$ , respectively, thus indicating a decided uniform tendency in the third differences. The first function in error is clearly the value for Feb. 7, and the last, that for Feb. 11. There may be an uncertainty of a unit or two in the values of their corrections at the outset; a few trials, however, will indicate that  $-3.2$  is the best value to apply to  $+0.1$  in  $\Delta'''$ , and  $+3.0$  to the term  $-11.8$ . By means of these corrections, the first three and the last two values of  $\Delta'''$  are brought into practical uniformity. In the column  $c_1 \text{ \& } c_2$  are given the corrections to  $\Delta'''$ , according to the binomial numbers, 1, 3, 3, 1. In the next column, the sum  $\Delta''' + c_1 + c_2$  is written, which evidently requires a third correction, tabulated under  $c_3$ .

The differences are now sufficiently smooth. Since  $c_3$  corresponds to a correction of  $-5''.1$  to  $\delta$  for Feb. 9, we conclude that the correct values of  $\delta$  for Feb. 7, 9, and 11, should read,  $-15^{\circ} 12' 9''.8$ ,  $-14^{\circ} 33' 59''.1$ , and  $-13^{\circ} 54' 49''.8$ , respectively.

It occasionally happens that some order of difference clearly

indicates a correction corresponding to the binomial coefficients of a lower order than that of the difference in question. This means the existence of an error in some earlier order of *difference*, rather than an error in the column of functions. For example, if  $\Delta^v$  requires a correction of the order 1, 3, 3, 1, it follows that an error exists in  $\Delta''$ , since  $\Delta^v$  is the *third* difference of  $\Delta''$ . More generally, when  $\Delta^{(n)}$  requires a correction according to the binomial coefficients of the  $m^{\text{th}}$  order, an error exists in  $\Delta^{(n-m)}$ . These remarks imply the necessity of some caution on the part of the beginner.

It will be observed that when either the first or last function of a series is in error, only the first or the last term in each order of difference will be affected, and only by an amount numerically equal to the error. Hence, in such cases, the method above explained is of little value.

In general, it may be stated that when errors have been discovered by differencing, it is advisable to re-compute the values in question, when the data for the calculation are available.

## GENERAL PROPERTIES OF DIFFERENCES.

11. Let  $F(t), F(t+\omega), F(t+2\omega), \dots$  represent any series of tabular functions, whose differences are taken as in the schedule below :

[illegible]

We shall assume that  $F(T)$  is a finite and continuous function, and that  $F(t+s\omega)$  is capable of expansion in a series of powers of  $s\omega$ , within the limits of the given table; then, denoting the successive derivatives of  $F(T)$  by  $F'(T)$ ,  $F''(T)$ , etc., we have, by TAYLOR'S Theorem, the following expressions :

$$\left. \begin{aligned} F(t) &= F(t) \\ F(t+\omega) &= F(t) + \omega F'(t) + \frac{\omega^2}{2} F''(t) + \frac{\omega^3}{6} F'''(t) + \frac{\omega^4}{24} F^{iv}(t) + \dots \\ F(t+2\omega) &= F(t) + 2\omega F'(t) + 4 \frac{\omega^2}{2} F''(t) + 8 \frac{\omega^3}{6} F'''(t) + 16 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ F(t+3\omega) &= F(t) + 3\omega F'(t) + 9 \frac{\omega^2}{2} F''(t) + 27 \frac{\omega^3}{6} F'''(t) + 81 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ F(t+4\omega) &= F(t) + 4\omega F'(t) + 16 \frac{\omega^2}{2} F''(t) + 64 \frac{\omega^3}{6} F'''(t) + 256 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ &\dots \end{aligned} \right\} \quad (0)$$

Differencing these values of the functions in the usual manner, we obtain successively the expressions for  $\Delta'$ ,  $\Delta''$ ,  $\Delta''' \dots$ , as follows :

$$\left. \begin{aligned} \Delta'_0 &= \omega F'(t) + \frac{\omega^2}{2} F''(t) + \frac{\omega^3}{6} F'''(t) + \frac{\omega^4}{24} F^{iv}(t) + \dots \\ \Delta'_1 &= \omega F'(t) + 3 \frac{\omega^2}{2} F''(t) + 7 \frac{\omega^3}{6} F'''(t) + 15 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ \Delta'_2 &= \omega F'(t) + 5 \frac{\omega^2}{2} F''(t) + 19 \frac{\omega^3}{6} F'''(t) + 65 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ \Delta'_3 &= \omega F'(t) + 7 \frac{\omega^2}{2} F''(t) + 37 \frac{\omega^3}{6} F'''(t) + 175 \frac{\omega^4}{24} F^{iv}(t) + \dots \\ &\dots \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} \Delta''_0 &= \omega^2 F''(t) + \omega^3 F'''(t) + \frac{7}{12} \omega^4 F^{iv}(t) + \dots \\ \Delta''_1 &= \omega^2 F''(t) + 2\omega^3 F'''(t) + \frac{2}{1} \frac{5}{2} \omega^4 F^{iv}(t) + \dots \\ \Delta''_2 &= \omega^2 F''(t) + 3\omega^3 F'''(t) + \frac{5}{1} \frac{5}{2} \omega^4 F^{iv}(t) + \dots \\ &\dots \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} \Delta'''_0 &= \omega^3 F'''(t) + \frac{3}{2} \omega^4 F^{iv}(t) + \dots \\ \Delta'''_1 &= \omega^3 F'''(t) + \frac{5}{2} \omega^4 F^{iv}(t) + \dots \\ &\dots \end{aligned} \right\} \quad (3)$$

It will be observed that all terms of the expansions (0) are of the general form,  $K\omega^r F^{(r)}(t)$ ; where  $K$  denotes a numerical factor, and  $r$  an integer which increases by unity as we proceed from any term to the next term following. Hence, the *differences* will contain

only terms of this form. We thus see, *a priori*, that any difference of the  $n^{\text{th}}$  order must be of the form

$$A_s^{(n)} = A\omega^r F^{(r)}(t) + B\omega^{r+1} F^{(r+1)}(t) + C\omega^{r+2} F^{(r+2)}(t) + D\omega^{r+3} F^{(r+3)}(t) + \dots$$

Let us now *assume* what appears from (1), (2), and (3) to be the general law; that is

$$A = 1 \qquad r = n$$

leaving the coefficients  $B, C, D, \dots$  undetermined for the present. We therefore assume

$$A_s^{(n)} = \omega^n F^{(n)}(t) + B\omega^{n+1} F^{(n+1)}(t) + C\omega^{n+2} F^{(n+2)}(t) + D\omega^{n+3} F^{(n+3)}(t) + \dots \quad (4)$$

Since the value of  $t$  is arbitrary, we may write  $t + \omega$  for  $t$ ; by making this substitution in the right-hand member of (4), we evidently get the expression for the  $n^{\text{th}}$  difference immediately following  $A_s^{(n)}$ , —that is, the value of  $A_{s+1}^{(n)}$ . Hence we have

$$A_{s+1}^{(n)} = \omega^n F^{(n)}(t + \omega) + B\omega^{n+1} F^{(n+1)}(t + \omega) + C\omega^{n+2} F^{(n+2)}(t + \omega) + D\omega^{n+3} F^{(n+3)}(t + \omega) + \dots$$

Developing the functions of the right-hand member by TAYLOR'S Theorem, we find

$$\begin{aligned} A_{s+1}^{(n)} = & \omega^n \left[ F^{(n)}(t) + \omega F^{(n+1)}(t) + \frac{\omega^2}{2} F^{(n+2)}(t) + \frac{\omega^3}{6} F^{(n+3)}(t) + \dots \right] \\ & + B\omega^{n+1} \left[ F^{(n+1)}(t) + \omega F^{(n+2)}(t) + \frac{\omega^2}{2} F^{(n+3)}(t) + \dots \right] \\ & + C\omega^{n+2} \left[ F^{(n+2)}(t) + \omega F^{(n+3)}(t) + \dots \right] \\ & + D\omega^{n+3} \left[ F^{(n+3)}(t) + \dots \right] + \dots \end{aligned}$$

Collecting the coefficients of  $F^{(n)}(t), F^{(n+1)}(t), \dots$ , we obtain

$$\begin{aligned} A_{s+1}^{(n)} = & \omega^n F^{(n)}(t) + (B+1)\omega^{n+1} F^{(n+1)}(t) + \left(C+B+\frac{1}{2}\right)\omega^{n+2} F^{(n+2)}(t) \\ & + \left(D+C+\frac{B}{2}+\frac{1}{6}\right)\omega^{n+3} F^{(n+3)}(t) + \dots \end{aligned} \quad (5)$$

Subtracting (4) from (5), and observing that  $A_{s+1}^{(n)} - A_s^{(n)} = A_s^{(n+1)}$ , we get

$$\begin{aligned} A_s^{(n+1)} = & \omega^{n+1} F^{(n+1)}(t) + \left(B+\frac{1}{2}\right)\omega^{n+2} F^{(n+2)}(t) + \left(C+\frac{B}{2}+\frac{1}{6}\right)\omega^{n+3} F^{(n+3)}(t) \\ & + \left(D+\frac{C}{2}+\frac{B}{6}+\frac{1}{24}\right)\omega^{n+4} F^{(n+4)}(t) + \dots \end{aligned}$$

If, therefore, we put

$$\left. \begin{aligned} B' &= B + \frac{1}{[2]} \\ C' &= C + \frac{B}{[2]} + \frac{1}{[3]} \\ D' &= D + \frac{C}{[2]} + \frac{B}{[3]} + \frac{1}{[4]} \\ &\dots\dots\dots \end{aligned} \right\} \tag{6}$$

we have

$$A_s^{(n+1)} = \omega^{n+1} F^{(n+1)}(t) + B' \omega^{n+2} F^{(n+2)}(t) + C' \omega^{n+3} F^{(n+3)}(t) + D' \omega^{n+4} F^{(n+4)}(t) + \dots\dots \tag{7}$$

Hence, if the general form of expression assumed in (4) is true for the index  $n$ , it follows from (7) that it is also true for  $n+1$ ; but we see by equations (1), (2), and (3), that the law obtains for  $n=1, 2, 3$ , respectively; hence it holds for  $n=4$ ; and so on indefinitely. The expression (4) is therefore true for all positive integral values of  $n$ .

12. We have now to determine the coefficients  $B, C, D, \dots\dots$ , of equation (4). These quantities are evidently functions of  $n$  and  $s$ , and will be determined in the following manner:

*First*, we take  $s=0$ , and determine the constants for  $A_0^{(n)}$ , which we shall denote for this purpose by  $B_n, C_n, D_n, \dots\dots$

These values are found by induction, thus: the relations (6) give  $B_{n+1}, C_{n+1}, D_{n+1}, \dots\dots$  in terms of  $B_n, C_n, D_n, \dots\dots$ . Making  $n=1$ , we take  $B_1, C_1, D_1, \dots\dots$  directly from the first of the equations (1); a continued application of (6) therefore gives successively the values of  $B_2, B_3, B_4, \dots\dots B_{n-1}, B_n$ . Similarly, we derive  $C_n, D_n, \dots\dots$ . Hence, the coefficients of (4) become known for  $s=0$ .

*Second*, the coefficients of  $A_s^{(n)}$  easily follow from those of  $A_0^{(n)}$ ; for it is clear from the schedule of §11 that  $A_s^{(n)}$  is related to  $F(t+s\omega)$  in precisely the manner that  $A_0^{(n)}$  is related to  $F(t)$ . Hence, if for brevity we write

$$A_0^{(n)} = \Psi(t)$$

we shall have, since the value of  $t$  is arbitrary,

$$A_s^{(n)} = \Psi(t+s\omega)$$

Then, expanding  $\Psi(t+s\omega)$  in a series of powers of  $s\omega$ , we arrive at an expression of the form (4), in which the coefficients are fully determined functions of  $n$  and  $s$ .

To perform the steps indicated, we take from the first of the equations (1) the following values:

$$B_1 = \frac{1}{2} \quad C_1 = \frac{1}{6} \quad D_1 = \frac{1}{24} \quad . \quad . \quad . \quad . \quad (8)$$

To find  $B_n$ : By repeated application of the first of (6), we have

$$\begin{aligned} B_2 &= B_1 + \frac{1}{2} \\ B_3 &= B_2 + \frac{1}{2} \\ B_4 &= B_3 + \frac{1}{2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ B_{n-1} &= B_{n-2} + \frac{1}{2} \\ B_n &= B_{n-1} + \frac{1}{2} \end{aligned}$$

Hence, by the addition of these  $n-1$  equations, we get

$$B_n = B_1 + \frac{1}{2}(n-1) = \frac{1}{2} + \frac{1}{2}(n-1) = \frac{n}{2} \quad (9)$$

To find  $C_n$ : Using the second of (6), we obtain

$$\begin{aligned} C_2 &= C_1 + \frac{1}{2} B_1 + \frac{1}{6} \\ C_3 &= C_2 + \frac{1}{2} B_2 + \frac{1}{6} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ C_n &= C_{n-1} + \frac{1}{2} B_{n-1} + \frac{1}{6} \end{aligned}$$

whence, by addition, we find

$$C_n = C_1 + \frac{1}{2}(B_1 + B_2 + \cdot \quad \cdot \quad \cdot \quad + B_{n-1}) + \frac{1}{6}(n-1)$$

Since  $C_1 = \frac{1}{6}$ , this gives

$$C_n = \frac{1}{2}(B_1 + B_2 + \cdot \quad \cdot \quad \cdot \quad + B_{n-1}) + \frac{n}{6} = \frac{1}{2} \sum_{r=1}^{n-1} B_r + \frac{n}{6}$$

But, from (9), we have  $B_r = \frac{r}{2}$ ; hence we get

$$C_n = \frac{1}{4} \sum_{r=1}^{n-1} r + \frac{n}{6} = \frac{1}{4} \left[ \frac{n(n-1)}{2} \right] + \frac{n}{6} = \frac{n}{24} (3n+1) \quad (10)$$

To find  $D_n$ : Again, from (6), we derive

$$\begin{aligned} D_2 &= D_1 + \frac{1}{2} C_1 + \frac{1}{6} B_1 + \frac{1}{24} \\ D_3 &= D_2 + \frac{1}{2} C_2 + \frac{1}{6} B_2 + \frac{1}{24} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ D_n &= D_{n-1} + \frac{1}{2} C_{n-1} + \frac{1}{6} B_{n-1} + \frac{1}{24} \end{aligned}$$

whence

$$D_n = D_1 + \frac{1}{2} \sum_{r=1}^{n-1} C_r + \frac{1}{6} \sum_{r=1}^{n-1} B_r + \frac{1}{24} (n-1) = \frac{1}{2} \sum_{r=1}^{n-1} C_r + \frac{n^2}{24}$$

From (10), we have

$$C_r = \frac{r}{24} (3r+1) = \frac{r^2}{8} + \frac{r}{24}$$

$$\therefore D_n = \frac{1}{16} \sum_{r=1}^{n-1} r^2 + \frac{1}{48} \sum_{r=1}^{n-1} r + \frac{n^2}{24} = \frac{1}{16} \left[ \frac{n}{6} (n-1)(2n-1) \right] + \frac{1}{48} \left[ \frac{n(n-1)}{2} \right] + \frac{n^2}{24}$$

$$\text{or} \quad D_n = \frac{n^2}{48} (n+1) \quad (11)$$

In like manner, the process might be extended to the values of  $E_n, F_n, \dots$ ; but the results already obtained are here sufficient. Substituting in equation (4) the values of  $B_n, C_n$ , and  $D_n$ , given by (9), (10), (11), (remembering that these values suppose  $s=0$ ), we have

$$A_0^{(n)} = \omega^n F^{(n)}(t) + \frac{n}{2} \omega^{n+1} F^{(n+1)}(t) + \frac{n}{24} (3n+1) \omega^{n+2} F^{(n+2)}(t) + \frac{n^2}{48} (n+1) \omega^{n+3} F^{(n+3)}(t) + \dots \quad (12)$$

We now obtain from (12) the expression for  $A_s^{(n)}$ . As already proposed, we write

$$A_0^{(n)} = \Psi(t) = \omega^n F^{(n)}(t) + B_n \omega^{n+1} F^{(n+1)}(t) + C_n \omega^{n+2} F^{(n+2)}(t) + \dots$$

Then, as shown above, we shall have

$$\begin{aligned} A_s^{(n)} &= \Psi(t+sw) = \Psi(t) + s\omega \Psi'(t) + \frac{s^2 \omega^2}{2} \Psi''(t) + \frac{s^3 \omega^3}{6} \Psi'''(t) + \dots \\ &= \left( \omega^n F^{(n)}(t) + B_n \omega^{n+1} F^{(n+1)}(t) + C_n \omega^{n+2} F^{(n+2)}(t) + D_n \omega^{n+3} F^{(n+3)}(t) + \dots \right) \\ &\quad + s\omega \left( \omega^n F^{(n+1)}(t) + B_n \omega^{n+1} F^{(n+2)}(t) + C_n \omega^{n+2} F^{(n+3)}(t) + \dots \right) \\ &\quad + \frac{s^2 \omega^2}{2} \left( \omega^n F^{(n+2)}(t) + B_n \omega^{n+1} F^{(n+3)}(t) + \dots \right) \\ &\quad + \frac{s^3 \omega^3}{6} \left( \omega^n F^{(n+3)}(t) + \dots \right) + \dots \end{aligned}$$

Upon arranging this expression according to ascending powers of  $\omega$ , we get

$$\begin{aligned} A_s^{(n)} &= \omega^n F^{(n)}(t) + (B_n + s) \omega^{n+1} F^{(n+1)}(t) + \left( C_n + B_n s + \frac{s^2}{2} \right) \omega^{n+2} F^{(n+2)}(t) \\ &\quad + \left( D_n + C_n s + \frac{B_n s^2}{2} + \frac{s^3}{6} \right) \omega^{n+3} F^{(n+3)}(t) + \dots \end{aligned} \quad (13)$$

Hence, substituting the foregoing values of  $B_n, C_n$ , and  $D_n$ , we

find that the values of  $B, C, D, \dots$  in equation (4) are as follows:

$$\left. \begin{aligned} B &= \frac{n}{2} + s \\ C &= \frac{n}{24} (3n+1) + \frac{s}{2} (n+s) \\ D &= \left( \frac{n+2s}{12} \right) \left[ \frac{n(n+1)}{4} + s(n+s) \right] \\ &\dots \dots \dots \end{aligned} \right\} \quad (14)$$

These results are easily verified by substituting special values of  $n$  and  $s$ , and comparing with the coefficients in equations (1), (2), (3); thus, putting  $s=1$ , and taking  $n=1, 2, 3$ , successively, we obtain the numerical coefficients in the expansions of  $A_1', A_1'',$  and  $A_1'''$ , respectively.

13. *Remarkable Formal Relation between the Expressions for  $A_0^{(n)}$  and  $A_0'$ .*—The coefficients  $B_n, C_n, D_n, \dots$ , in the expression for  $A_0^{(n)}$ , may also be determined by the following method, which not only is shorter than the above, but also possesses the advantage of showing a direct relation between the expressions for  $A_0^{(n)}$  and  $A_0'$ , respectively. Retaining the above notation, we write (12) in the form

$$A_0^{(n)} = \omega^n F^{(n)}(t) + B_n \omega^{n+1} F^{(n+1)}(t) + C_n \omega^{n+2} F^{(n+2)}(t) + \dots \quad (15)$$

We now let

$$\varphi_n(y) \equiv y^n + B_n y^{n+1} + C_n y^{n+2} + D_n y^{n+3} + \dots \quad (15a)$$

be an auxiliary expression, such that the coefficient of  $y^{n+r}$  is the coefficient of  $\omega^{n+r} F^{(n+r)}(t)$  in (15). Writing  $n+1$  for  $n$  in (15a), and using the relations (6), we have

$$\begin{aligned} \varphi_{n+1}(y) &= y^{n+1} + \left( B_n + \frac{1}{2} \right) y^{n+2} + \left( C_n + \frac{B_n}{2} + \frac{1}{3} \right) y^{n+3} \\ &\quad + \left( D_n + \frac{C_n}{2} + \frac{B_n}{3} + \frac{1}{4} \right) y^{n+4} + \dots \end{aligned} \quad (16)$$

Again, since the coefficients of  $\varphi_1(y)$  are those of  $A_0'$ , we obtain from (1),

$$\varphi_1(y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \dots \quad (17)$$

By re-arranging the terms of (16), we find

$$\begin{aligned}
 \varphi_{n+1}(y) &= y^{n+1} + \frac{y^{n+2}}{\underline{2}} + \frac{y^{n+3}}{\underline{3}} + \frac{y^{n+4}}{\underline{4}} + \dots \\
 &\quad + B_n \left( y^{n+2} + \frac{y^{n+3}}{\underline{2}} + \frac{y^{n+4}}{\underline{3}} + \frac{y^{n+5}}{\underline{4}} + \dots \right) \\
 &\quad + C_n \left( y^{n+3} + \frac{y^{n+4}}{\underline{2}} + \frac{y^{n+5}}{\underline{3}} + \frac{y^{n+6}}{\underline{4}} + \dots \right) \\
 &\quad + D_n \left( y^{n+4} + \frac{y^{n+5}}{\underline{2}} + \frac{y^{n+6}}{\underline{3}} + \frac{y^{n+7}}{\underline{4}} + \dots \right) \\
 &\quad + \dots \\
 &= y^n \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right) \\
 &\quad + B_n y^{n+1} \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right) \\
 &\quad + C_n y^{n+2} \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right) \\
 &\quad + D_n y^{n+3} \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right) \\
 &\quad + \dots \\
 &= \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right) \left( y^n + B_n y^{n+1} + C_n y^{n+2} + D_n y^{n+3} + \dots \right)
 \end{aligned}$$

Hence, by (15a) and (17), we have

$$\varphi_{n+1} = \varphi_1 \cdot \varphi_n$$

Taking  $n = 1, 2, 3, \dots, n-1$ , successively, we find

$$\begin{array}{ll}
 \varphi_2 = \varphi_1 \varphi_1 & \varphi_6 = \varphi_1 \varphi_5 \\
 \varphi_3 = \varphi_1 \varphi_2 & \dots \\
 \varphi_4 = \varphi_1 \varphi_3 & \varphi_{n-1} = \varphi_1 \varphi_{n-2} \\
 \varphi_5 = \varphi_1 \varphi_4 & \varphi_n = \varphi_1 \varphi_{n-1}
 \end{array}$$

Multiplying these equations together member for member, and cancelling the common factors, we obtain

$$\varphi_n = (\varphi_1)^n \quad (18)$$

Therefore, by (17), we have

$$\begin{aligned}
 \varphi_n(y) &= \left( y + \frac{y^2}{\underline{2}} + \frac{y^3}{\underline{3}} + \frac{y^4}{\underline{4}} + \dots \right)^n = y^n \left( 1 + \frac{y}{\underline{2}} + \frac{y^2}{\underline{3}} + \frac{y^3}{\underline{4}} + \dots \right)^n \\
 \therefore \varphi_n(y) &= y^n + \frac{n}{2} y^{n+1} + \frac{n}{24} (3n+1) y^{n+2} + \frac{n^2}{48} (n+1) y^{n+3} + \dots \quad (19)
 \end{aligned}$$

Comparing coefficients in (15a) and (19), we find

$$B_n = \frac{n}{2} \quad , \quad C_n = \frac{n}{24} (3n+1) \quad , \quad D_n = \frac{n^2}{48} (n+1), \quad . . . . \quad (20)$$

Substituting these values in (15), the latter becomes

$$\Delta_0^{(n)} = \omega^n F^{(n)}(t) + \frac{n}{2} \omega^{n+1} F^{(n+1)}(t) + \frac{n}{24} (3n+1) \omega^{n+2} F^{(n+2)}(t) + \frac{n^2}{48} (n+1) \omega^{n+3} F^{(n+3)}(t) + . .$$

which agrees with (12).

These results may be conveniently expressed symbolically: thus, let us represent the quantities  $\Delta_0', \Delta_0'', \Delta_0''', . . . \Delta_0^{(n)}$  by  $\Delta_0, \Delta_0^2, \Delta_0^3, . . . \Delta_0^n$ ; and for  $\omega F'(t), \omega^2 F''(t), \omega^3 F'''(t), . . . \omega^n F^{(n)}(t)$  let us write the symbols  $D, D^2, D^3, . . . . D^n$ , respectively; then we shall have

$$\left. \begin{aligned} \Delta_0 &= D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24} + \frac{D^5}{120} + . . . . \\ \Delta_0^2 &= \left( D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24} + . . . . \right)^2 = D^2 + D^3 + \frac{7}{12} D^4 + \frac{1}{4} D^5 + . . . . \\ \Delta_0^3 &= \left( D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24} + . . . . \right)^3 = D^3 + \frac{3}{2} D^4 + \frac{5}{4} D^5 + \frac{3}{4} D^6 + . . . . \\ . . . . . \\ \Delta_0^n &= \left( D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24} + . . . . \right)^n \\ &= D^n + \frac{n}{2} D^{n+1} + \frac{n}{24} (3n+1) D^{n+2} + \frac{n^2}{48} (n+1) D^{n+3} + . . . . \end{aligned} \right\} \quad (21)$$

14. THEOREM V.—*The  $n^{\text{th}}$  differences of any rational integral expression of the  $n^{\text{th}}$  degree are constant. If the general form of the function is  $F(T) \equiv \alpha T^n + \beta T^{n-1} + \gamma T^{n-2} + . . . .$ , the constant value of  $\Delta^{(n)}$  is  $\omega^n \alpha \lfloor n$ .*

For, from the nature of the function, we have, evidently,

$$F^{(n)}(t) = \frac{d^{(n)} F}{dT^n} = \alpha \lfloor n$$

and

$$F^{(n+1)}(t) = F^{(n+2)}(t) = . . . . = 0$$

Hence, from (4), we have

$$\Delta_s^{(n)} = \omega^n F^{(n)}(t) = \omega^n \alpha \lfloor n \quad (22)$$

The theorem is therefore true, whatever the value of the constant interval  $\omega$ . Several examples have already occurred: in §2 we have

the differences of  $F(T) \equiv T^4 - 10T^2 - 20$ ; here  $n = 4$ ,  $a = 1$ ,  $\omega = 1$ . Hence, by (22), we get

$$\Delta^v = \underline{14} = 24$$

—the value already found by differencing.

In Example I of §9,  $F(T) \equiv T^3$ ,  $\omega = 1$ ; we there obtained for the value of the third difference

$$\Delta''' = 6$$

which agrees with the theorem.

Again, in Example I of §10,  $F(T) \equiv 2T^3 - 25T - 40$ ,  $\omega = 1$ ; whence the theorem requires

$$\Delta''' = a\underline{12} = 2\underline{12} = 12$$

which is the result already obtained.

15. THEOREM VI.—*If the  $n^{\text{th}}$  differences of a series of quantities (tabulated for equidistant values of  $T$ ) are constant, the given quantities are the tabular values of a rational integral function of the form  $F(T) \equiv aT^n + \beta T^{n-1} + \gamma T^{n-2} + \dots$*

This proposition is the converse of THEOREM V, and is proved as follows:

Let  $F(T)$  denote the function whose *true mathematical values*, tabulated for the given values of  $T$ , form the given series of quantities. From (4) and (5), we see that the expressions for  $\Delta_s^{(n)}$  and  $\Delta_{s+1}^{(n)}$  agree only in their first term,  $\omega^n F^{(n)}(t)$ ; the remaining terms of like order in  $\omega$  having unlike coefficients. Hence, the conditions necessary in order that  $\Delta^{(n)}$  shall be constant throughout are as follows:

*First*, that  $\omega^n F^{(n)}(t)$  does *not* vanish;

*Second*, that  $\omega^{n+1} F^{(n+1)}(t) = \omega^{n+2} F^{(n+2)}(t) = \dots = 0$ ;

But, since  $\omega$  cannot vanish, these conditions reduce to the form—

$$\left. \begin{aligned} F^{(n)}(t) &\geq 0 \\ F^{(n+1)}(t) &= F^{(n+2)}(t) = \dots = 0 \end{aligned} \right\} \quad (23)$$

If now we put

$$T = t + \tau \quad (24)$$

then, by TAYLOR'S Theorem, we have

$$F(T) = F(t+\tau) = F(t) + \tau F'(t) + \frac{\tau^2}{2} F''(t) + \dots + \frac{\tau^n}{n!} F^{(n)}(t) + \frac{\tau^{n+1}}{(n+1)!} F^{(n+1)}(t) + \dots$$

By (23), this gives

$$F(T) = F(t) + \tau F'(t) + \dots + \frac{\tau^{n-1}}{(n-1)!} F^{(n-1)}(t) + \frac{\tau^n}{n!} F^{(n)}(t) \quad (25)$$

in which, we observe, the coefficient of  $\tau^n$  cannot vanish. Substituting in (25) the value of  $\tau$  given by (24), we obtain

$$F(T) = F(t) + (T-t) F'(t) + (T-t)^2 \frac{F''(t)}{2} + \dots + (T-t)^n \frac{F^{(n)}(t)}{n!}$$

Since  $t$  has a fixed value, the right-hand member of this equation is an expression of the  $n^{\text{th}}$  degree in the variable  $T$ , and hence may be written in the form

$$F(T) \equiv \alpha T^n + \beta T^{n-1} + \gamma T^{n-2} + \dots \quad (26)$$

which establishes the theorem.

16. *Convergence of the Differences in Practice.*—In the discussion of Theorems V and VI, we were concerned with the true mathematical values of the quantities involved. In practice, however, the absolute or true mathematical values of functions are seldom employed; frequently, as previously noted, a function is tabulated only to a certain degree of approximation, enough decimals being retained to give the desired accuracy. We observe that in such cases there is a tendency of the differences to decrease numerically, and usually to vanish sensibly, as the order of difference progresses. The explanation of this tendency follows readily from equation (4), thus: for any given function, the derivatives  $F^{(n)}(t)$ ,  $F^{(n+1)}(t)$ ,  $F^{(n+2)}(t)$ ,  $\dots$  have definite values; hence, the value of  $\omega$  may be chosen sufficiently small to render all the terms in the second member of (4) insensible, *except the first*. When this condition obtains, the value of  $\Delta^{(n)}$  is sensibly constant, and equal to  $\omega^n F^{(n)}(t)$ . The differences of  $F(T)$  are thus practically brought to a termination at the  $n^{\text{th}}$  order, whether the function is algebraic or transcendental.

In many cases the values of the successive derivatives converge rapidly; the chosen value of  $\omega$  may then be quite large, and yet allow the differences to sensibly vanish at an early order. This is equivalent

to the obvious statement that, when a function is to be tabulated so as to difference readily, the interval of the argument must be decided by the manner in which the given function varies.

To exemplify these principles, we take the following table of seven-figure logarithms:

$T$	$\text{Log } T$	$\Delta'$	$\Delta''$	$\Delta'''$
1.00	0.0000000	+43214		
1.01	.0043214	42788	−426	+8
1.02	.0086002	42370	418	9
1.03	.0128372	41961	409	8
1.04	.0170333	41560	401	+7
1.05	.0211893	−394		
1.06	0.0253059	+41166		

In this case,  $\omega = 0.01$ ,  $t = 1.00$ ,  $t + \omega = 1.01$ ,  $t + 2\omega = 1.02$ , etc. To serve our present purpose, we here transcribe from (1), (2), and (3), the following expressions:

$$\left. \begin{aligned} \Delta'_0 &= \omega F'(t) + \frac{\omega^2}{2} F''(t) + \frac{\omega^3}{6} F'''(t) + \frac{\omega^4}{24} F^{iv}(t) + \dots \\ \Delta''_0 &= \omega^2 F''(t) + \omega^3 F'''(t) + \frac{7}{12} \omega^4 F^{iv}(t) + \dots \\ \Delta'''_0 &= \omega^3 F'''(t) + \frac{3}{2} \omega^4 F^{iv}(t) + \dots \\ \Delta^{iv}_0 &= \omega^4 F^{iv}(t) + \dots \end{aligned} \right\} \tag{27}$$

Since  $F(T) = \log T$ , we have

$$F'(t) = +Mt^{-1}, \quad F''(t) = -Mt^{-2}, \quad F'''(t) = +2Mt^{-3}, \quad F^{iv}(t) = -6Mt^{-4}, \quad \dots$$

where  $M$  is the modulus of the common system of logarithms,  $= 0.434294$ . Hence, with  $t = 1$  and  $\omega = 0.01$ , we find

$$\begin{aligned} \omega F'(t) &= +0.0043429,4 & \omega^3 F'''(t) &= +0.0000008,7 \\ \omega^2 F''(t) &= -0.0000434,3 & \omega^4 F^{iv}(t) &= -0.0000000,3 \end{aligned}$$

Substituting these numerical values in (27), we obtain, in units of the 7th decimal,

$$\Delta'_0 = +43214 \qquad \Delta''_0 = -426 \qquad \Delta'''_0 = +8 \qquad \Delta^{iv}_0 = 0$$

which agree substantially with the results obtained above by direct differencing. The rapid convergence of the differences is thus seen

to be due to the small value of the interval  $\omega$ , which makes the term  $\omega^3 F'''(t)$  appreciable, but renders  $\omega^4 F^{iv}(t)$ ,  $\omega^5 F^v(t)$ , . . . quite insensible; accordingly,  $\Delta'''$  is the last difference which we need take into account, the remaining differences being practically zero.

We may add that if the values of  $T$  in the present table were 100, 101, etc., instead of the given values, the interval  $\omega$  would become 1 instead of 0.01, and hence  $\omega, \omega^2, \omega^3, \omega^4, \dots$  would not converge as above. We should then, however, have  $t=100$  instead of 1, which would cause the successive *derivatives* to converge rapidly, as is obvious from the general expression

$$F^{(n)}(t) = (-1)^{n-1} M \underline{n-1} \cdot \frac{1}{t^n}$$

Furthermore, the differences of  $F(T)$  contain only terms of the form

$$K\omega^n F^{(n)}(t) = (-1)^{n-1} KM \underline{n-1} \left(\frac{\omega}{t}\right)^n$$

where  $K$  denotes a numerical factor; hence, since the values of  $\omega$  and  $t$  are both increased one hundred-fold by the assumed change, it is evident that the general term  $K\omega^n F^{(n)}(t)$  is not altered thereby. The *differences* are therefore unaltered by the proposed change; this conclusion is confirmed by the consideration that the assumed alteration in  $T$  would merely change the logarithmic characteristic from 0 to 2, and thus would not affect the resulting differences. These observations illustrate the case of a tabular function whose successive derivatives converge rapidly, whereby a comparatively large argument interval may be used, and yet allow the resulting series of differences to converge as rapidly as may be required.

17. As a second example, we consider the following table of cubes:

$T$	$T^3$	$\Delta'$	$\Delta''$	$\Delta'''$
5.16	137.39			
5.21	141.42	+4.03	+0.08	0
5.26	145.53	4.11	.08	0
5.31	149.72	4.19	.08	0
5.36	153.99	4.27	.08	0
5.41	158.34	4.35	+0.08	
5.46	162.77	+4.43		

We have already seen (Theorem V) that when the true mathematical values of  $T^3$  are tabulated, the third differences are constant, the fourth differences being the first order to vanish. In the present table, however, only two decimals have been retained in  $T^3$ , whereas the true value involves six places. To this degree of approximation, the third differences are entirely insensible; this follows from Theorem V, which gives for the constant value of  $\Delta'''$ —

$$\Delta''' = \omega^3 \alpha \lfloor 3$$

In this example we have

$$\omega = 0.05 \qquad \alpha = 1$$

and hence

$$\Delta''' = (0.05)^3 \times 6 = 0.00,075$$

which is insensible when only two decimals are concerned. Thus, in the approximations so frequently used in practice, the differences generally terminate (either absolutely or approximately) at some order earlier than would occur if the true mathematical values of the function were employed.

It may be added that the above example affords an illustration of Theorem VI. For, since the second differences are here absolutely constant, it follows from this theorem that the tabular quantities are the true mathematical values (corresponding to the given values of  $T$ ) of some function of the form

$$F(T) \equiv \alpha T^2 + \beta T + \gamma$$

Thus, in particular, if the student tabulates the function

$$F(T) \equiv 16(T^2 - 5.3325T + 9.476975)$$

for  $T = 5.16, 5.21, \dots, 5.46$ , and retains all decimals involved, he will find his tabular numbers identical with the above series.

18. *To Express  $\omega^n F^{(n)}(t)$  in Terms of  $\Delta_0^{(n)}, \Delta_0^{(n+1)}, \Delta_0^{(n+2)}$ , etc.*—The problem consists in reversing the series (15), which expresses  $\Delta_0^{(n)}$  in terms of  $\omega^n F^{(n)}(t), \omega^{n+1} F^{(n+1)}(t), \dots$

Let us denote  $\omega^r F^{(r)}(t)$  by  $x_r$ ; then, writing successively,  $n, n+1, n+2, \dots$  for  $n$  in (15), we have

$$\left. \begin{aligned} A_0^{(n)} &= x_n + B_n x_{n+1} + C_n x_{n+2} + D_n x_{n+3} + \dots \\ A_0^{(n+1)} &= x_{n+1} + B_{n+1} x_{n+2} + C_{n+1} x_{n+3} + D_{n+1} x_{n+4} + \dots \\ A_0^{(n+2)} &= x_{n+2} + B_{n+2} x_{n+3} + C_{n+2} x_{n+4} + D_{n+2} x_{n+5} + \dots \\ &\dots \end{aligned} \right\} \quad (28)$$

from which we obtain, by transposition,

$$\left. \begin{aligned} x_n &= A_0^{(n)} - B_n x_{n+1} - C_n x_{n+2} - D_n x_{n+3} - \dots \\ x_{n+1} &= A_0^{(n+1)} - B_{n+1} x_{n+2} - C_{n+1} x_{n+3} - D_{n+1} x_{n+4} - \dots \\ x_{n+2} &= A_0^{(n+2)} - B_{n+2} x_{n+3} - C_{n+2} x_{n+4} - D_{n+2} x_{n+5} - \dots \\ &\dots \end{aligned} \right\} \quad (29)$$

The second of the equations (29) gives a value of  $x_{n+1}$ , which, substituted in the first equation, gives  $x_n$  in terms of  $A_0^{(n)}, A_0^{(n+1)}, x_{n+2}, x_{n+3}, \dots$ ; substituting in the latter expression the value of  $x_{n+2}$  given by the third of (29), we find  $x_n$  in terms of  $A_0^{(n)}, A_0^{(n+1)}, A_0^{(n+2)}, x_{n+3}, x_{n+4}, \dots$ . Continuing this process of elimination indefinitely, we arrive at an expression of the form

$$x_n \equiv \omega^n F^{(n)}(t) = A_0^{(n)} + b_n A_0^{(n+1)} + c_n A_0^{(n+2)} + d_n A_0^{(n+3)} + \dots \quad (30)$$

The coefficients  $b_n, c_n, d_n, \dots$  must now be determined. From (15a) we obtain the following group of equations:

$$\left. \begin{aligned} q_n &= y^n + B_n y^{n+1} + C_n y^{n+2} + D_n y^{n+3} + \dots \\ q_{n+1} &= y^{n+1} + B_{n+1} y^{n+2} + C_{n+1} y^{n+3} + D_{n+1} y^{n+4} + \dots \\ q_{n+2} &= y^{n+2} + B_{n+2} y^{n+3} + C_{n+2} y^{n+4} + D_{n+2} y^{n+5} + \dots \\ &\dots \end{aligned} \right\} \quad (31)$$

Comparing (28) and (31), we observe that the latter group may be obtained from the former by writing  $q_r$  and  $y^r$  for  $A_0^{(r)}$  and  $x_r$ , respectively; the algebraic relations in both groups are otherwise identical. Hence, if from (31) we seek to express  $y^n$  in terms of  $q_n, q_{n+1}, q_{n+2}, \dots$ , the process of reversion will be identical with that which gives  $x_n$  in terms of  $A_0^{(n)}, A_0^{(n+1)}, \dots$ ; hence we must find

$$y^n = q_n + b_n q_{n+1} + c_n q_{n+2} + d_n q_{n+3} + \dots \quad (32)$$

the coefficients being those of (30). Therefore, by (18), we have

$$y^n = q_1^n + b_n q_1^{n+1} + c_n q_1^{n+2} + d_n q_1^{n+3} + \dots \quad (33)$$

Taking  $n = 1$ , in (30) and (33), we obtain

$$x_1 = A_0' + b_1 A_0'' + c_1 A_0''' + d_1 A_0^{iv} + \dots \quad (34)$$

and

$$y = q_1 + b_1 q_1^2 + c_1 q_1^3 + d_1 q_1^4 + \dots \quad (35)$$



that a second tabulation of  $F(T)$  has been made, differing from the first only in the value of the interval,  $\omega$ . Let  $\omega' = m\omega$  be the interval of the argument in the second table; denoting the differences by  $\delta', \delta'', \delta''', \dots$ , the new table will run as follows:

$T$	$F(T)$	$\delta'$	$\delta''$	$\delta'''$	$\delta^{iv}$	$\dots$
$t$	$F(t)$					
$t + m\omega$	$F(t + m\omega)$	$\delta_0'$	$\delta_0''$			
$t + 2m\omega$	$F(t + 2m\omega)$	$\delta_1'$	$\delta_1''$	$\delta_0'''$	$\delta_0^{iv}$	
$t + 3m\omega$	$F(t + 3m\omega)$	$\delta_2'$	$\delta_2''$	$\delta_1'''$	$\delta_1^{iv}$	$\dots$
$t + 4m\omega$	$F(t + 4m\omega)$	$\delta_3'$	$\delta_3''$	$\delta_2'''$	$\delta_2^{iv}$	$\dots$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

We proceed to investigate the relations between  $\delta', \delta'', \delta''', \dots$  and  $\delta', \delta'', \delta''', \dots$ . No restriction is placed upon the value of  $m$ ; in the applications of the resulting formulae, however,  $m$  will usually be regarded as a positive proper fraction. The second tabulation will then give the function for closer values of  $T$  than the first.

Since the value of  $\omega$  is arbitrary, we may write  $m\omega$  for  $\omega$  in the right-hand member of (15), and thus obtain the expression for  $\delta_0^{(n)}$ ; making this substitution, we find

$$\delta_0^{(n)} = m^n \omega^n F^{(n)}(t) + B_n m^{n+1} \omega^{n+1} F^{(n+1)}(t) + C_n m^{n+2} \omega^{n+2} F^{(n+2)}(t) + \dots \tag{46}$$

If, as above, we write  $x_r$  for  $\omega^r F^{(r)}(t)$ , this equation becomes

$$\delta_0^{(n)} = m^n x_n + B_n m^{n+1} x_{n+1} + C_n m^{n+2} x_{n+2} + \dots \tag{47}$$

From (30) we obtain, in succession,

$$\left. \begin{aligned} x_n &= A_0^{(n)} + b_n A_0^{(n+1)} + c_n A_0^{(n+2)} + d_n A_0^{(n+3)} + \dots \\ x_{n+1} &= A_0^{(n+1)} + b_{n+1} A_0^{(n+2)} + c_{n+1} A_0^{(n+3)} + d_{n+1} A_0^{(n+4)} + \dots \\ &\dots \end{aligned} \right\} \tag{48}$$

Eliminating  $x_n, x_{n+1}, \dots$  from (47), by means of (48), there results an equation of the form

$$\delta_0^{(n)} = m^n A_0^{(n)} + \beta_n A_0^{(n+1)} + \gamma_n A_0^{(n+2)} + \dots \tag{49}$$

which, for  $n = 1$ , becomes

$$\delta_0' = m A_0' + \beta_1 A_0'' + \gamma_1 A_0''' + \dots \tag{50}$$

Now let

$$z_n \equiv m^n y^n + B_n m^{n+1} y^{n+1} + C_n m^{n+2} y^{n+2} + \dots \quad (51)$$

be an auxiliary expression, such that the coefficient of  $y^{n+r}$  is the coefficient of  $x_{n+r}$  in (47).

From (33) we obtain, in succession,

$$\left. \begin{aligned} y^n &= \varphi_1^n + b_n \varphi_1^{n+1} + c_n \varphi_1^{n+2} + d_n \varphi_1^{n+3} + \dots \\ y^{n+1} &= \varphi_1^{n+1} + b_{n+1} \varphi_1^{n+2} + c_{n+1} \varphi_1^{n+3} + d_{n+1} \varphi_1^{n+4} + \dots \\ &\dots \dots \dots \end{aligned} \right\} \quad (52)$$

Now, to eliminate  $y^n, y^{n+1}, \dots$  from (51), by means of (52), we must perform precisely the same algebraic steps as in the derivation of equation (49) from (47) and (48); we shall therefore obtain

$$z_n = m^n \varphi_1^n + \beta_n \varphi_1^{n+1} + \gamma_n \varphi_1^{n+2} + \dots \quad (53)$$

and, for  $n = 1$ , we have

$$z_1 = m \varphi_1 + \beta_1 \varphi_1^2 + \gamma_1 \varphi_1^3 + \dots \quad (54)$$

Now the equation (51) may be written

$$z_n = (my)^n + B_n (my)^{n+1} + C_n (my)^{n+2} + \dots$$

Whence, by (15a), we have

$$z_n = \varphi_n(my) \quad (55)$$

and hence, also,

$$z_1 = \varphi_1(my)$$

or, by (17),

$$\begin{aligned} z_1 &= (my) + \frac{(my)^2}{1^2} + \frac{(my)^3}{1^3} + \frac{(my)^4}{1^4} + \dots = e^{my} - 1 \\ \therefore 1 + z_1 &= e^{my} \end{aligned} \quad (56)$$

Also, from (36), we have

$$1 + \varphi_1 = e^y$$

the combination of which with (56) gives

$$1 + z_1 = (1 + \varphi_1)^m = 1 + m \varphi_1 + \frac{m(m-1)}{1^2} \varphi_1^2 + \dots + \frac{m(m-1) \dots (m-r+1)}{1^r} \varphi_1^r + \dots$$

or

$$z_1 = m \varphi_1 + \frac{m(m-1)}{1^2} \varphi_1^2 + \dots + \frac{m(m-1) \dots (m-r+1)}{1^r} \varphi_1^r + \dots \quad (57)$$

Comparing (54) and (57), we find

$$\beta_1 = \frac{m(m-1)}{1^2} \quad , \quad \gamma_1 = \frac{m(m-1)(m-2)}{1^3} \quad , \quad . . . . . \quad (58)$$

Substituting these values in (50), we obtain the following fundamental relation:

$$\delta_0' = m\Delta_0' + \frac{m(m-1)}{1^2}\Delta_0'' + . . . . + \frac{m(m-1) . . . . (m-r+1)}{1^r}\Delta_0^{(r)} + . . . . \quad (59)$$

Again, using the relation  $\varphi_n = \varphi_1^n$ , we obtain from (55),

$$z_n = \varphi_n(my) = \{\varphi_1(my)\}^n = z_1^n \quad (60)$$

Hence, from (57), we find

$$z_n = \left( m\varphi_1 + \frac{m(m-1)}{1^2}\varphi_1^2 + \frac{m(m-1)(m-2)}{1^3}\varphi_1^3 + . . . . \right)^n$$

Expanding and factoring, we obtain

$$\begin{aligned} z_n &= m^n \varphi_1^n + \frac{n}{2} m^n (m-1) \varphi_1^{n+1} + \frac{n}{24} \left[ (3n+1)m - (3n+5) \right] m^n (m-1) \varphi_1^{n+2} \\ &+ \frac{n}{48} \left[ n(n+1)m^2 - 2(n^2+3n+1)m + (n+2)(n+3) \right] m^n (m-1) \varphi_1^{n+3} + . . . \end{aligned} \quad (61)$$

Equating coefficients of like powers of  $\varphi_1$  in (53) and (61), we have

$$\beta_n = \frac{n}{2} m^n (m-1) \quad , \quad \gamma_n = \frac{n}{24} m^n (m-1) \left[ (3n+1)m - (3n+5) \right] \quad , \quad . . . . \quad (62)$$

Substituting these values in (49), the latter becomes

$$\begin{aligned} \delta_0^{(n)} &= m^n \Delta_0^{(n)} + \frac{n}{2} m^n (m-1) \Delta_0^{(n+1)} + \frac{n}{24} m^n (m-1) \left[ (3n+1)m - (3n+5) \right] \Delta_0^{(n+2)} \\ &+ \frac{n}{48} m^n (m-1) \left[ n(n+1)m^2 - 2(n^2+3n+1)m + (n+2)(n+3) \right] \Delta_0^{(n+3)} + . . . . \end{aligned} \quad (63)$$

Finally, we may symbolize these results by the following expressions: (64)

$$\begin{aligned} \delta_0 &= m\Delta_0 + \frac{m(m-1)}{1^2}\Delta_0^2 + \frac{m(m-1)(m-2)}{1^3}\Delta_0^3 + \frac{m(m-1) . . (m-3)}{1^4}\Delta_0^4 + \frac{m(m-1) . . (m-4)}{1^5}\Delta_0^5 + . . \\ \delta_0^2 &= \left( m\Delta_0 + \frac{m(m-1)}{1^2}\Delta_0^2 + \frac{m(m-1)(m-2)}{1^3}\Delta_0^3 + . . . . \right)^2 \\ &= m^2\Delta_0^2 + m^2(m-1)\Delta_0^3 + \frac{m^2}{12}(m-1)(7m-11)\Delta_0^4 + \frac{m^2}{12}(m-1)(m-2)(3m-5)\Delta_0^5 + . . . . \\ \delta_0^3 &= \left( m\Delta_0 + \frac{m(m-1)}{1^2}\Delta_0^2 + . . . . \right)^3 = m^3\Delta_0^3 + \frac{3}{2}m^3(m-1)\Delta_0^4 + \frac{m^3}{4}(m-1)(5m-7)\Delta_0^5 + . . . . \\ \delta_0^4 &= \left( m\Delta_0 + \frac{m(m-1)}{1^2}\Delta_0^2 + . . . . \right)^4 = m^4\Delta_0^4 + 2m^4(m-1)\Delta_0^5 + \frac{m^4}{6}(m-1)(13m-17)\Delta_0^6 + . . . . \\ \delta_0^5 &= \left( m\Delta_0 + \frac{m(m-1)}{1^2}\Delta_0^2 + . . . . \right)^5 = m^5\Delta_0^5 + \frac{5}{2}m^5(m-1)\Delta_0^6 + \frac{5}{8}m^5(m-1)(4m-5)\Delta_0^7 + . . . . \\ &. . . . . \end{aligned}$$

20. THEOREM VII.—*If the  $n^{\text{th}}$  differences of a given series of functions are numerically large as compared with all the following differences, then, if the series be re-tabulated with the argument interval  $m$  times its original value, the  $n^{\text{th}}$  differences of the new series will be approximately  $m^n$  times the corresponding  $n^{\text{th}}$  differences of the original series.*

The theorem is a direct interpretation of equation (63). For, if  $\Delta_0^{(n+1)}, \Delta_0^{(n+2)}, \dots$  are all small in comparison with  $\Delta_0^{(n)}$ , then the approximate value of  $\delta_0^{(n)}$  is  $m^n \Delta_0^{(n)}$ .

COROLLARY.—*If the  $n^{\text{th}}$  differences of the given series are constant, then the  $n^{\text{th}}$  differences of the new series are also constant, and equal to  $m^n$  times the original  $n^{\text{th}}$  differences.*

For, if  $\Delta^{(n)}$  is constant,  $\Delta^{(n+1)}, \Delta^{(n+2)}, \dots$  are all zero, and hence (63) gives, rigorously,

$$\delta^{(n)} = m^n \Delta^{(n)}$$

21. To illustrate the foregoing results, we take the following table of cubes:

$T$	$F(T) \equiv T^3$	$\Delta'$	$\Delta''$	$\Delta'''$
100	1000000			
103	1092727	+ 92727		
106	1191016	98289	+5562	+162
109	1295029	104013	5724	162
112	1404928	109899	5886	+162
115	1520875	+115947	+6048	

Here the interval  $\omega = 3$ . If we take  $m = \frac{1}{3}$ , the interval is reduced to 1, and hence the new table is as follows:

$T$	$T^3$	$\delta'$	$\delta''$	$\delta'''$
100	1000000			
101	1030301	+30301		
102	1061208	30907	+606	+6
103	1092727	31519	612	6
104	1124864	32137	618	+6
105	1157625	+32761	+624	

We now test the first three of the equations (64); substituting

in the latter  $m = \frac{1}{3}$ , and observing that the differences beyond  $\Delta'''$  vanish, we find

$$\delta_0' = \frac{1}{3}\Delta_0' - \frac{1}{9}\Delta_0'' + \frac{5}{81}\Delta_0''' \quad , \quad \delta_0'' = \frac{1}{9}\Delta_0'' - \frac{2}{27}\Delta_0''' \quad , \quad \delta_0''' = \frac{1}{27}\Delta_0''' \quad (65)$$

From the first of the above tables, we take

$$\Delta_0' = +92727 \quad \Delta_0'' = +5562 \quad \Delta_0''' = +162$$

Whence, from (65), we derive

$$\delta_0' = 30909 - 618 + 10 = 30301 \quad \delta_0'' = 618 - 12 = 606 \quad \delta_0''' = 6$$

which agree exactly with the values found in the second table above. It will be observed that  $\delta_0'$  and  $\delta_0''$  come within  $\frac{1}{30}$  part of equaling  $\frac{1}{3}\Delta_0'$  and  $\frac{1}{9}\Delta_0''$ , respectively; while  $\delta_0''' = \frac{1}{27}\Delta_0'''$ , exactly. These relations are in accord with Theorem VII.

22. *To Express the Differences of  $F(T)$  in Terms of the Given Functions only.*—Let the given series be  $F_0, F_1, F_2, F_3, \dots$ ; then the first differences are  $F_1 - F_0, F_2 - F_1, F_3 - F_2, \dots$ ; the second differences,  $F_2 - 2F_1 + F_0, F_3 - 2F_2 + F_1, \dots$ ; the third differences,  $F_3 - 3F_2 + 3F_1 - F_0, F_4 - 3F_3 + 3F_2 - F_1, \dots$ ; and so on. The coefficients evidently follow the *binomial law*. Thus we have generally

$$\Delta_0^{(n)} = F_n - nF_{n-1} + \frac{n(n-1)}{2}F_{n-2} - \dots + (-1)^r {}_nC_r F_{n-r} \pm \dots + (-1)^{n-1} nF_1 + (-1)^n F_0 \quad (66)$$

in which, according to the usual notation, we put  ${}_nC_r$  for the coefficient of  $x^r$  in the expansion of  $(1+x)^n$ .

To prove (66), let us assume it true for the index  $n$ ; then the expression for the  $n^{\text{th}}$  difference immediately following  $\Delta_0^{(n)}$  (*i.e.*,  $\Delta_1^{(n)}$ ) will be obtained by increasing the subscripts of  $F_n, F_{n-1}, \dots$  in (66) by unity. We therefore have

$$\Delta_1^{(n)} = F_{n+1} - nF_n + \frac{n(n-1)}{2}F_{n-1} - \dots + (-1)^{r+1} {}_nC_{r+1} F_{n-r} \pm \dots + (-1)^n F_1 \quad (67)$$

Subtracting (66) from (67), we find

$$\begin{aligned} \Delta_0^{(n+1)} = \Delta_1^{(n)} - \Delta_0^{(n)} &= F_{n+1} - (n+1)F_n + \frac{(n+1)n}{2}F_{n-1} - \dots \\ &\quad + (-1)^{r+1} ({}_nC_{r+1} + {}_nC_r) F_{n-r} \pm \dots + (-1)^n (n+1)F_1 + (-1)^{n+1} F_0 \end{aligned}$$

But, as proved in Algebra, we have

$${}_nC_{r+1} + {}_nC_r = {}_{n+1}C_{r+1}$$

and hence the preceding equation becomes

(68)

$$\Delta_0^{(n+1)} = F_{n+1} - (n+1)F_n + \frac{(n+1)n}{2}F_{n-1} - \dots + (-1)^{r+1}{}_nC_{r+1}F_{n-r} \pm \dots + (-1)^{n+1}F_0$$

It follows from (68) that if the law expressed in (66) holds for  $n$ , it also holds for  $n+1$ . But we have seen above that the expression is true for  $n=1, 2$  and  $3$ . Hence it is true for  $n=4$ , and so on indefinitely; the equation (66) is therefore true for all positive integral values of  $n$ .

23. *To Express Any Function of a Given Series in Terms of Some Particular Function ( $F_0$ ), and of the Differences ( $a_0, b_0, c_0, \dots$ ) which Follow that Function.*—As before, let  $F_0, F_1, F_2, F_3, \dots$  denote the given series, the differences being taken as in the schedule below:

$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$
$F_0$	$a_0$	$b_0$				
$F_1$	$a_1$	$b_1$	$c_0$	$d_0$		
$F_2$	$a_2$	$b_2$	$c_1$	$d_1$	$e_0$	$f_0$
$F_3$	$a_3$	$b_3$	$c_2$	$d_2$	$e_1$	$f_1$
$F_4$	$a_4$		$c_3$		$e_2$	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$F_{n-1}$		$b_{n-2}$	.	.	.	.
$F_n$	$a_{n-1}$	$b_{n-1}$	.	.	.	.
$F_{n+1}$	$a_n$	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

Let it be required to express  $F_n$  in terms of  $F_0, a_0, b_0, c_0, d_0, \dots$ . From the nature of the differences, we have

$$\begin{aligned} F_1 &= F_0 + a_0 \\ F_2 &= F_1 + a_1 = (F_0 + a_0) + (a_0 + b_0) = F_0 + 2a_0 + b_0 \\ F_3 &= F_2 + a_2 = (F_0 + 2a_0 + b_0) + (a_0 + 2b_0 + c_0) = F_0 + 3a_0 + 3b_0 + c_0 \end{aligned}$$

and so on. The coefficients again follow the binomial law, which suggests for the form of the general term —

$$F_n = F_0 + na_0 + \frac{n(n-1)}{[2]} b_0 + \frac{n(n-1)(n-2)}{[3]} c_0 + \dots \quad (69)$$

To prove (69) by induction, we assume that it is true for the index  $n$ . Moreover, we evidently have

$$F_{n+1} = F_n + a_n$$

We may now find  $a_n$  in terms of  $a_0, b_0, c_0, d_0, \dots$  from (69), — since the relation is here the same as the relation of  $F_n$  to  $F_0, a_0, b_0, c_0, \dots$ ; thus we obtain

$$a_n = a_0 + nb_0 + \frac{n(n-1)}{[2]} c_0 + \dots$$

Adding this value of  $a_n$  to that of  $F_n$  given by (69), we find\*

$$F_{n+1} = F_n + a_n = F_0 + (n+1)a_0 + \frac{(n+1)n}{[2]} b_0 + \frac{(n+1)n(n-1)}{[3]} c_0 + \dots \quad (70)$$

Thus, having assumed the relation (69) to be true for the index  $n$ , we find by (70) that it is also true when  $n+1$  is written for  $n$ ; but we have shown directly that (69) holds for  $n = 1, 2$  and  $3$ . The formula (69) is therefore true for all positive integral values of  $n$ .

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\* We here omit the proof for the general term, since the process is the same as in § 22.

EXAMPLES.

1. Tabulate the five-place logarithms of 25, 30, 35, . . . . 65, 70, and take the differences to the fifth order inclusive. Retain a copy of the table for further use.
2. Tabulate  $F(T) \equiv \log \cos T$ , to five decimals, for  $T = 50^\circ, 53^\circ, 56^\circ, \dots 74^\circ, 77^\circ$ ; difference to the fifth order, as in Example 1. Retain a copy of the table.
3. Verify the accuracy of both the functions and their differences in Examples 1 and 2, by noting the degree of regularity in  $\Delta^n$ , according to the method of §8.
4. Also, rigorously check the differencing in the above examples, by taking the algebraic sum of each separate order, as explained in §3.
5. Add the two series of functions tabulated in Examples 1 and 2; difference the new series as before, and see that the resulting values of  $\Delta^n$  are the sums of the fifth differences of the other series, according to Theorem IV.
6. Correct the errors in the following tables by the method of differences:

(a)		(b)		(c)	
$T$	$F(T) \equiv \frac{1}{T}$	Appa. Alt. of Star	Mean Refraction	Latitude	Reduction
0.21	4.762	10°	5' 19.2"	0°	0' 0.00"
.23	4.348	12	4' 27.5	2	0' 48.02
.25	4.000	14	3' 49.5	4	1' 35.80
.27	3.704	16	3' 18.4	6	2' 23.12
.29	3.465	18	2' 57.5	8	3' 9.75
.31	3.226	20	2' 38.8	10	3' 55.11
.33	3.030	22	2' 23.3	12	4' 40.05
.35	2.857	24	2' 10.2	14	5' 23.28
.37	2.703	26	1' 58.9	16	6' 4.95
0.39	2.564			18	6' 44.86

(d)		(e)		(f)	
$T$	$F(T) \equiv T \sin r$	Date 1898	Log. Dist. of <i>Mars from Earth</i>	Date 1898	Lunar Dist. of <i>Jupiter</i>
0.48	0.7125	Sept. 17	0.139162	Dec. 1.0	105° 5' 59"
.50	.7173	21	.130819	1.5	99 18 28
.52	.7226	25	.122145	2.0	93 31 31
.54	.7273	29	.113130	2.5	87 44 46
.56	.7349	Oct. 3	.103759	3.0	81 57 48
.58	.7419	7	.094015	3.5	76 10' 17
.60	.7494	11	.083857	4.0	70 21 14
.62	.7568	15	.073360	4.5	64 30 37
.64	.7660	19	.062478	5.0	58 39 44
.66	.7751	23	.051135	5.5	52 42 5
.68	.7847	27	.039438	6.0	46 43 12
.70	.7947	31	.027351	6.5	40 40 43
0.72	0.8052	Nov. 4	0.014875	7.0	34 34 29

7. Tabulate the following rational integral functions for the assigned values of the argument. Before taking the differences, state at which order the latter become constant, and compute the constant value in each case, by Theorem V. Then take the differences, and see that the results agree with the computed values.

- (a)  $F(T) \equiv T^6 - 50T^4 + 100T^2$ .  
(Tabulate for  $T = -8, -6, -4, -2, 0, +2, +4, +6, +8$ .)
- (b)  $F(T) \equiv 2T^3 - 7T - 400$ . ( $T = 8.0, 8.3, 8.6, . . . 9.8$ .)
- (c)  $F(T) \equiv 0.16T^4 - 0.3T^3$ . ( $T = 2, 3, 4, 5, 6, 7, 8$ .)

8. By means of the first of equations (1), compute the value of  $\mathcal{A}'$  which immediately follows  $\log \cos 56^\circ$  in the table of Example 2. The value of  $\omega (= 3^\circ)$  must be expressed in circular measure. Compare the computed with the tabular value.

9. Tabulate  $F(T) \equiv \log T$ , to five places of decimals, for  $T = 30, 40, 50, 60, 70$ ; denote this table by  $B$ , and that of Example 1 by  $A$ .  $A$  and  $B$  then differ only in  $\omega$ , the interval having now been doubled. Then, in the second of the equations (64), put  $m = 2$ , and substitute from  $A$  the values of  $\mathcal{A}_0'', \mathcal{A}_0''', \mathcal{A}_0^{iv}$ , and  $\mathcal{A}_0^v$ , which correspond to  $T = 40$ . Whence, compute the value of  $\delta_0''$  corresponding to  $T = 40$  in  $B$ , and compare computed with actual value.

10. In Example 1, compute the quantities  $\mathcal{A}_0^{iv}$  and  $F_5 (= \log 50)$ , by (66) and (69) respectively; compare the results with the values found in the table.

## CHAPTER II.

### OF INTERPOLATION.

24. *Statement of the Problem.*—Given a series of numerical values of a function, for equidistant values of the argument, it is required to find the value of the function for *any intermediate* value of the argument, independently of the analytical form of the function, which may or may not be given.

*Interpolation* is the process or method by which the required values are found.

Without certain restrictions or assumptions as to the character of the function and the interval of its tabulation, the problem of interpolation is an indeterminate one. Thus it is evident, *a priori*, that from a series of temperatures recorded for every noon at a given station, it would be impossible to obtain by interpolation the temperature at 8.00 P.M., for a given day. If, *per contra*, the thermometric readings were recorded for 7.00, 7.10, 7.20, 7.30, . . . . P.M., it is highly probable that the temperature at 7.14 P.M. could be interpolated with accuracy.

The *Nautical Almanac* gives the heliocentric longitude of *Jupiter* for every 4th day; but, because of the slow, continuous, and systematic character of *Jupiter's* orbital motion, it is found sufficient to compute the longitudes from the tables direct for every 40th day only. The intermediate places are then readily interpolated with an accuracy which equals, if indeed it does not exceed, that of direct computation.

The moon's longitude is given in the *Nautical Almanac* for every twelve hours; for the moon's orbital motion is so rapid and complicated that it would prove inexpedient to attempt the interpolation of accurate values of the longitude from an ephemeris given for whole day intervals.

It therefore appears that, to render the problem of interpolation determinate, the tabular interval ( $\omega$ ) must be sufficiently small that the nature or law of the function will be definitively shown by the tabular values in question. The condition thus imposed will be satisfied when, in a given table, the differences become either *rigorously* or *sensibly* constant at some particular order.\* This follows from the fact, soon to be proved, that for all such cases a formula of interpolation can be established, either *rigorously* or *sensibly* true, according to the foregoing distinction.

25. *Extension of Formula (69) to Fractional and Negative Values of  $n$ , Provided the Differences of Some Particular Order are Constant.*—We have shown (Theorem V) that the differences of a rational integral function vanish beyond a certain order. We proceed to prove that, for any such function, the formula (69) is rigorously true for *all* values of  $n$ .

Let  $F(T)$  denote any function whose differences become constant at the order  $i$ , and let  $\Delta^{(i)} = l_0$ ;  $F(T)$  and its differences are then shown in the schedule on following page.

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\*Excepting, of course, any *periodic* function whose tabular interval ( $\omega$ ) differs but little from some multiple of its period,  $P$ . An example of such a series is the following :

Date, 1898	Day of the Year	Heliocentric Longitude of Mercury	$\Delta'$	$\Delta''$	$\Delta'''$
		$\begin{smallmatrix} 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 \end{smallmatrix}$
Jan. 4	4	93 0	+12 33	—26	
Apr. 4	94	105 33	12 7	33	—7
July 3	184	117 40	11 34	—38	—5
Oct. 1	274	129 14	+10 56		
Dec. 30	364	140 10			

where  $P$  (the time of one revolution of *Mercury*) = 87.97 days; and hence  $\omega = 90^d = P + 2^d.03$ . The differences  $\Delta'$  therefore correspond to a tabular interval of 2.03 days, and *not* to the interval 90 days, as the table itself would indicate. Now, the actual value of *Mercury's* longitude for Jan. 14 is found from the *Nautical Almanac* to be  $l = 149^\circ 40'$ ; if, however, we fail to account for the periodic character of this function, and argue solely from the numerical data at hand, we find by a rough interpolation, for Jan. 14,

$$l = 93.0 + \left(\frac{1}{90} \times 12.6\right) = 94.4$$

which bears no relation to the truth. The possibility of thus committing serious error through failing to account for completed periods or revolutions, suggests the necessity of caution in this direction.

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\dots$	$\Delta^{(i)}$
$t$	$F_0$				
$t+\omega$	$F_1$	$a_0$	$b_0$		
$t+2\omega$	$F_2$	$a_1$	$b_1$	$\dots$	$l_0$
$t+3\omega$	$F_3$	$a_2$	$b_2$	$\dots$	$l_0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$l_0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$l_0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$l_0$
$t+(i+2)\omega$	$F_{i+2}$		$b_{i+1}$	$\dots$	
$t+(i+3)\omega$	$F_{i+3}$	$a_{i+2}$			

From (30) we obtain, in succession,

$$\begin{aligned}\omega^i F^{(i)}(t) &= \Delta_0^{(i)} + b_i \Delta_0^{(i+1)} + c_i \Delta_0^{(i+2)} + \dots \\ \omega^{i+1} F^{(i+1)}(t) &= \Delta_0^{(i+1)} + b_{i+1} \Delta_0^{(i+2)} + \dots \\ \omega^{i+2} F^{(i+2)}(t) &= \Delta_0^{(i+2)} + \dots \\ &\dots\end{aligned}$$

With the condition assumed, these equations give

$$\begin{aligned}\omega^i F^{(i)}(t) &= l_0 \\ \omega^{i+1} F^{(i+1)}(t) &= \omega^{i+2} F^{(i+2)}(t) = \dots = 0\end{aligned}$$

Hence, in this case, the expansions (0) end at the  $(i+1)$ th term. It follows that, under the present assumption, the expansions (0) are valid; in other words,  $F(t+n\omega)$  is capable of expansion by TAYLOR'S Theorem for all values of  $n$  within the limits of the given table. Hence, for *all* such values, we have

$$F_n \equiv F(t+n\omega) = F(t) + n\omega F'(t) + \frac{n^2\omega^2}{\lfloor 2} F''(t) + \dots + \frac{n^i\omega^i}{\lfloor i} F^{(i)}(t) \tag{71}$$

Let us now consider the expression

$$Q \equiv F_0 + na_0 + \frac{n(n-1)}{\lfloor 2} b_0 + \dots + \frac{n(n-1) \dots (n-i+1)}{\lfloor i} l_0 \tag{72}$$

Substituting, successively,  $n = 0, 1, 2, 3, \dots, i+3$ , in (72), we get, according to (69),

$$Q = F_0, F_1, F_2, F_3, \dots, F_{i+3}, \text{ respectively.}$$

Substituting these same values of  $n$  in (71), we evidently obtain the same results, namely —

$$F_n = F_0, F_1, F_2, F_3, \dots, F_{i+3}, \text{ in succession.}$$

Hence,  $F_n$  and  $Q$  are equal to each other for more than  $i$  values of  $n$ . But  $F_n$  and  $Q$  are both expressions of the degree  $i$  in  $n$ . Now, when two expressions of the degree  $i$  in  $n$  are equal to each other for more than  $i$  values of  $n$ , they are equal for all values of  $n$ . Therefore, for *all* values of  $n$ , fractional and negative, we have

$$F_n \equiv F(t+n\omega) = F_0 + n\Delta_0' + \frac{n(n-1)}{1 \cdot 2} \Delta_0'' + \dots + \frac{n(n-1) \dots (n-i+1)}{1 \cdot 2 \dots i} \Delta_0^{(i)} \quad (73)$$

provided that  $\Delta^{(i)} = l_0 = \text{constant}$ . This is the fundamental formula of interpolation, and is known as NEWTON'S *Formula*.

26. *Second Proof of NEWTON'S Formula, for Constant Values of  $\Delta^{(i)}$ .*—Formula (73) is readily proved by means of equation (59), in which  $m$  may have any value. The only condition necessary for the validity of (59) is that the expansions (0) are themselves valid. But since we assume that the differences beyond  $\Delta^{(i)}$  vanish, it follows (as proved in the last section) that the expansions (0) *are* valid. Hence (59) gives, rigorously,

$$\delta_0' = m\Delta_0' + \frac{m(m-1)}{1 \cdot 2} \Delta_0'' + \dots + \frac{m(m-1) \dots (m-i+1)}{1 \cdot 2 \dots i} \Delta_0^{(i)}$$

From the definition of  $\delta_0'$  (see schedule, p. 31), we have

$$\begin{aligned} \delta_0' &= F(t+m\omega) - F(t) = F_m - F_0 \\ \therefore F_m \equiv F(t+m\omega) &= F_0 + \delta_0' \\ &= F_0 + m\Delta_0' + \frac{m(m-1)}{1 \cdot 2} \Delta_0'' + \dots + \frac{m(m-1) \dots (m-i+1)}{1 \cdot 2 \dots i} \Delta_0^{(i)} \end{aligned}$$

which is the same as formula (73), except that  $m$  is written for  $n$ .

27. *To Find  $n$ , the Interval of Interpolation.*—The binomial coefficients of NEWTON'S Formula are given in Table I, for every hundredth part of a unit in the argument  $n$ . The quantity  $n$  is called the *interval of interpolation*, and in practice is always less than unity. To obtain an expression for  $n$ , suppose that we are to interpolate the value of the function corresponding to the argument  $T$ , whose value lies between  $t$  and  $t+\omega$ ; then we shall have

$$F_n \equiv F(t+n\omega) = F(T) \quad , \quad \text{or} \quad t+n\omega = T$$

and therefore

$$n = \frac{T-t}{\omega} \quad (74)$$

which determines the interval  $n$ .

28. EXAMPLE.—From the following table of  $T^4$ , find the value of  $(2.8)^4$  by NEWTON'S Formula:

$T$	$F(T) \equiv T^4$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
2	16					
4	256	+ 240				
6	1296	1040	+ 800			
8	4096	2800	1760	+ 960	+384	
10	10000	5904	3104	1344	384	0
12	20736	10736	4832	1728	+384	0
14	38416	+17680	+6944	+2112		

Here we have

$$\begin{array}{ll}
 T = 2.8 & a_0 = +240 \\
 t = 2 & b_0 = +800 \\
 \omega = 2 & c_0 = +960 \\
 n = \frac{2.8-2}{2} = 0.4 & d_0 = +384 \\
 F_0 = 16 & e_0 = 0
 \end{array}$$

It will be convenient to denote the coefficients of  $a_0, b_0, c_0, \dots$  in (73) by  $A, B, C, \dots$ , respectively. Then, from Table I (with argument  $n = 0.40$ ), or by direct computation, we find

$$\begin{array}{ll}
 A = +0.40 & C = +0.0640 \\
 B = -0.12 & D = -0.0416
 \end{array}$$

We therefore obtain

$$\begin{array}{rcl}
 F_0 & = & +16.00 \\
 Aa_0 & = & +96.00 \\
 Bb_0 & = & -96.00 \\
 Cc_0 & = & +61.44 \\
 Dd_0 & = & -15.9744 \\
 \hline
 \therefore (2.8)^4 = F_{0.4} & = & +61.4656
 \end{array}$$

This result is easily verified, and found exact to the last figure. However, since Table I does not in general give the exact mathematical values of the interpolating coefficients, it follows that functions interpolated in this manner cannot always be *absolutely* correct. The results may be, as in logarithmic computation, but close approximations to the truth.

29. *Backward Interpolation.*—When the interval of interpolation approaches unity, it is usually more convenient to proceed *backwards* from the function which *follows* the value sought. The problem,

therefore, is to find  $F_{-n}$ ; for this purpose, let  $F(T)$  be differenced as in the schedule below—the values of  $\Delta^{(i)}$  being supposed constant as before:

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	. . .	$\Delta^{(i)}$
$t-3\omega$	$F_{-3}$		$b_{-4}$		$d_{-5}$		$l_0$
$t-2\omega$	$F_{-2}$	$a_{-3}$	$b_{-3}$	$c_{-4}$	$d_{-4}$	. . .	$l_0$
$t-\omega$	$F_{-1}$	$a_{-2}$	$b_{-2}$	$c_{-3}$	$d_{-3}$	. . .	$l_0$
$t$	$F_0$	$a_{-1}$	$b_{-1}$	$c_{-2}$	$d_{-2}$	. . .	$l_0$
$t+\omega$	$F_1$	$a_0$	$b_0$	$c_{-1}$	$d_{-1}$	. . .	$l_0$
$t+2\omega$	$F_2$	$a_1$	$b_1$	$c_0$	$d_0$	. . .	$l_0$
$t+3\omega$	$F_3$	$a_2$	$b_2$	$c_1$	$d_1$	. . .	$l_0$

We might substitute  $-n$  for  $n$  in (73), and find directly,

$$F_{-n} = F_0 + (-n)a_0 + \frac{(-n)(-n-1)}{[2]}b_0 + \frac{(-n)(-n-1)(-n-2)}{[3]}c_0 + \dots$$

But this formula, while true, is inconvenient from the fact that its coefficients neither converge as rapidly as the binomial coefficients for  $+n$ , nor can their numerical values be taken from Table I. To avoid the negative interval, we have only to suppose the series inverted, thus making  $F_3$  the first, and  $F_{-3}$  the last of the tabular functions. Then, by Theorem III, the signs of  $\Delta'$ ,  $\Delta'''$ ,  $\Delta^v, \dots$  are changed, while the signs of  $\Delta''$ ,  $\Delta^{iv}, \dots$  are unaltered. Now the value of  $F_{-n}$  is obtained by interpolating *forward* with the interval  $+n$  in the *inverted* series; hence the differences to be used in NEWTON'S Formula are—

$$-a_{-1}, +b_{-2}, -c_{-3}, +d_{-4}, \dots$$

We therefore have, by (73), (75)

$$F_{-n} \equiv F(t-n\omega) = F_0 - na_{-1} + \frac{n(n-1)}{[2]}b_{-2} - \frac{n(n-1)(n-2)}{[3]}c_{-3} + \frac{n(n-1)(n-2)(n-3)}{[4]}d_{-4} - \dots$$

the differences being taken as in the above schedule. The coefficients, as before, are taken from Table I with the argument  $n$ .

An immediate and important application of (75) is in finding the value of a function near the end of a given series. Thus, in the preceding schedule, suppose the series ended with  $F_0$ , and it were required to interpolate a value of  $F$  between  $F_{-1}$  and  $F_0$ : since the differences  $b_{-1}, c_{-1}, d_{-1}, \dots$  (required in interpolating forward from  $F_{-1}$ ) are not

given in this case, the formula (75) must be used;  $n$  being the interval of the required function from  $F_0$  toward  $F_{-1}$ .

EXAMPLE.—From the table of  $T^4$  given on page 44, find the value of  $(13.26)^4$ .

Taking  $t = 14$ , we find

$$n = \frac{14 - 13.26}{2} = 0.37$$

which is the interval counted *backwards* from  $F = 38416$ . Hence, from Table I, we obtain

$$\begin{array}{ll} A = +0.37 & C = +0.06333 \\ B = -0.11655 & D = -0.04164 \end{array}$$

And for the differences required by (75), we have

$$\begin{array}{ll} a_{-1} = +17680 & c_{-3} = +2112 \\ b_{-2} = +6944 & d_{-4} = +384 \end{array}$$

Therefore, by (75), we derive

$$\begin{array}{rcl} F_0 & = & +38416.00 \\ -Aa_{-1} & = & -6541.60 \\ +Bb_{-2} & = & -809.32 \\ -Cc_{-3} & = & -133.75 \\ +Dd_{-4} & = & -15.99 \\ \hline \therefore F_n = (13.26)^4 & = & +30915.34 \end{array}$$

By direct calculation, we find

$$(13.26)^4 = 30915.34492 +$$

30. *Application of NEWTON'S Formula, when the Differences Become only Approximately Constant.*—We have proved (§§25 and 26) that (73) is true for all values of  $n$ , provided the differences of some particular order are *rigorously* constant. We now propose to show that, if the value of  $n$  lies between 0 and  $+1$ , the formula is very approximately true for the more frequent case in which the differences of some order become *approximately*, but not absolutely constant. The example given on page 8 is typical of this case; the numbers involved are not the true mathematical values of the quantities represented, and hence the irregularities, as already explained.

Let  $F_0, F_1, F_2, F_3, \dots, F_r, \dots$  denote a series of approximate tabular values of any function  $F(T)$ , given for equidistant

values of  $T$ , and true to the *nearest* unit of their last figure; let  $\overline{F}_0, \overline{F}_1, \overline{F}_2, \overline{F}_3, \dots \overline{F}_r, \dots$  denote the corresponding true mathematical values of the series, which we shall designate generally as  $\overline{F}$ ; also, let  $F_r = \overline{F}_r + f_r$ ;  $f_r$  being the difference between the true and approximate values, due to the omission of decimals in the tabular quantities.

The differences of  $F$ , and those of the series  $f_0, f_1, f_2, f_3, \dots$ , are now defined by the two schedules below:

$T$	$\overline{F}(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\dots$	$\Delta^{(i)}$	$\Delta^{(i+1)}$	$\dots$
$t$	$\overline{F}_0$							
$t + \omega$	$\overline{F}_1$	$a_0$	$b_0$	$c_0$				
$t + 2\omega$	$\overline{F}_2$	$a_1$	$b_1$		$\dots$	$l_0$	$m_0$	
$t + 3\omega$	$\overline{F}_3$	$a_2$	$b_2$	$c_1$	$\dots$	$l_1$	$m_1$	$\dots$
$t + 4\omega$	$\overline{F}_4$	$a_3$	$b_3$	$c_2$	$\dots$	$l_2$	$m_2$	$\dots$
$t + 5\omega$	$\overline{F}_5$	$a_4$	$b_4$	$c_3$	$\dots$			$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

(A)

$T$	$f$	$\Delta'$	$\Delta''$	$\Delta'''$	$\dots$	$\Delta^{(i)}$	$\Delta^{(i+1)}$	$\dots$
$t$	$f_0$							
$t + \omega$	$f_1$	$\alpha_0$	$\beta_0$					
$t + 2\omega$	$f_2$	$\alpha_1$	$\beta_1$	$\gamma_0$	$\dots$	$\lambda_0$	$\mu_0$	
$t + 3\omega$	$f_3$	$\alpha_2$	$\beta_2$	$\gamma_1$	$\dots$	$\lambda_1$	$\mu_1$	$\dots$
$t + 4\omega$	$f_4$	$\alpha_3$	$\beta_3$	$\gamma_2$	$\dots$	$\lambda_2$	$\mu_2$	$\dots$
$t + 5\omega$	$f_5$	$\alpha_4$	$\beta_4$	$\gamma_3$	$\dots$			$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

(B)

Then, since  $F = \overline{F} + f$ , it follows from Theorem IV that the differences of  $F$  are as given in the appended table :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\dots$	$\Delta^{(i)}$	$\Delta^{(i+1)}$	$\dots$
$t$	$F_0 = \overline{F}_0 + f_0$							
$t + \omega$	$F_1 = \overline{F}_1 + f_1$	$a_0 + \alpha_0$	$b_0 + \beta_0$	$c_0 + \gamma_0$				
$t + 2\omega$	$F_2 = \overline{F}_2 + f_2$	$a_1 + \alpha_1$	$b_1 + \beta_1$	$c_1 + \gamma_1$	$\dots$	$l_0 + \lambda_0$		
$t + 3\omega$	$F_3 = \overline{F}_3 + f_3$	$a_2 + \alpha_2$	$b_2 + \beta_2$	$c_2 + \gamma_2$	$\dots$	$l_1 + \lambda_1$	$m_0 + \mu_0$	$\dots$
$t + 4\omega$	$F_4 = \overline{F}_4 + f_4$	$a_3 + \alpha_3$	$b_3 + \beta_3$	$c_3 + \gamma_3$	$\dots$	$l_2 + \lambda_2$	$m_1 + \mu_1$	$\dots$
$t + 5\omega$	$F_5 = \overline{F}_5 + f_5$	$a_4 + \alpha_4$	$b_4 + \beta_4$		$\dots$		$m_2 + \mu_2$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

(C)

Let us now suppose that the differences  $\Delta^{(i+1)}$  in Table (C) are either alternately  $+$  and  $-$ , or that  $+$  and  $-$  signs follow each other

irregularly. Moreover, the foregoing definition of  $F$  requires that the terms in  $\Delta^{(i+1)}$  are sufficiently small to indicate that no errors exceeding half a unit in the last place exist in the functions  $F(T)$ . The values of  $\Delta^{(i)}$  are then approximately constant, and therefore Table (C) represents the typical case in practice. We proceed to investigate the accuracy of NEWTON'S Formula as applied in this case; assuming that  $n$  is always taken within the limits 0 and  $+1$ , and that terms beyond  $\Delta^{(i)}$  are neglected.

Applying (73) to find  $F_n$  from Table (C), and omitting the terms beyond  $\Delta^{(i)}$ , we have

$$F_n = (\bar{F}_0 + f_0) + A(a_0 + \alpha_0) + B(b_0 + \beta_0) + C(c_0 + \gamma_0) + \dots + L(l_0 + \lambda_0) \quad (76)$$

in which  $A, B, C, \dots, L$  denote the binomial coefficients of the  $n$ th order. Let us now examine the approximate formula (76), to discover its *maximum error when all conditions conspire to that end*.

The formula (76) may be written

$$F_n = (\bar{F}_0 + A\alpha_0 + B\beta_0 + \dots + Ll_0) + (f_0 + A\alpha_0 + B\beta_0 + \dots + L\lambda_0) \quad (77)$$

For brevity, let us put

$$\left. \begin{aligned} Q &\equiv \bar{F}_0 + A\alpha_0 + B\beta_0 + \dots + Ll_0 \\ R &\equiv f_0 + A\alpha_0 + B\beta_0 + \dots + L\lambda_0 \\ \therefore F_n &= Q + R \end{aligned} \right\} \quad (77a)$$

It will be observed that  $Q$  is the value obtained for  $\bar{F}_n$  when (73) is applied to Table (A), terms beyond  $\Delta^{(i)}$  being neglected. We leave the discussion of  $Q$  for the present, to consider the quantity  $R$ , which evidently expresses the error of interpolation due to the unavoidable errors,  $f$ , contained in the tabular functions  $F$ .

Applying the formulae of §22 to the differences of Table (B), we have

$$\left. \begin{aligned} \alpha_0 &= f_1 - f_0 \\ \beta_0 &= f_2 - 2f_1 + f_0 \\ \gamma_0 &= f_3 - 3f_2 + 3f_1 - f_0 \\ \delta_0 &= f_4 - 4f_3 + 6f_2 - 4f_1 + f_0 \\ \epsilon_0 &= f_5 - 5f_4 + 10f_3 - 10f_2 + 5f_1 - f_0 \\ &\dots \dots \dots \end{aligned} \right\} \quad (78)$$

Hence, from (77a), we obtain

$$\begin{aligned}
 R &= f_0 + A\alpha_0 + B\beta_0 + C\gamma_0 + D\delta_0 + E\epsilon_0 + \dots + L\lambda_0 \\
 &= f_0 + A(f_1 - f_0) + B(f_2 - 2f_1 + f_0) + C(f_3 - 3f_2 + 3f_1 - f_0) \\
 &\quad + D(f_4 - 4f_3 + 6f_2 - 4f_1 + f_0) + \dots \\
 \therefore R &= f_0(1 - A + B - C + D - E + \dots \pm L) + f_1(A - 2B + 3C - 4D + 5E - \dots) \\
 &\quad + f_2(B - 3C + 6D - 10E + \dots) + f_3(C - 4D + 10E - \dots) \\
 &\quad + f_4(D - 5E + \dots) + f_5(E - \dots) + \dots
 \end{aligned} \tag{79}$$

Now the binomial coefficients  $A, B, C, \dots$  are connected by the following relations:

$$A = n, \quad B = \left(\frac{n-1}{2}\right)A, \quad C = \left(\frac{n-2}{3}\right)B, \quad \dots$$

Hence, since we have assumed that  $n$  lies between 0 and  $+1$ , it follows that  $A, B, C, \dots$  are alternately positive and negative, thus:

$$\begin{array}{ccccccccc}
 A & B & C & D & E & . & . & . & . \\
 + & - & + & - & + & . & . & . & .
 \end{array}$$

We therefore draw the following conclusions respecting (79):

$$\begin{array}{llll}
 \text{The coefficient of } f_1 & \text{is} & +; \\
 \text{" " " } f_2 & \text{"} & -; \\
 \text{" " " } f_3 & \text{"} & +; \\
 \text{" " " } f_4 & \text{"} & -; \\
 \dots & & \dots
 \end{array}$$

Now, since the values of  $F$  are supposed true to the *nearest* unit of the last decimal figure, the quantities  $f$  may have any value between  $-0.5$  and  $+0.5$ , in terms of the same unit; hence, it follows from the foregoing conclusions that if we take

$$f_1 = +0.5 \quad f_2 = -0.5 \quad f_3 = +0.5 \quad f_4 = -0.5 \quad \dots \tag{80}$$

the sum of all the terms *after the first* in the right-hand member of (79) will be numerically a *maximum*, with the  $+$  sign.

We shall now show that the coefficient of  $f_0$  in (79) is a positive number. For this purpose, let us consider the identity

$$(1-x)^{-1}(1-x)^n \equiv (1-x)^{n-1}$$

which, for all values of  $x$  numerically less than unity, may be expanded into the form

$$(1+x+x^2+x^3+\dots+x^i+\dots)(1-Ax+Bx^2-Cx^3+\dots\pm Lx^i\mp\dots) \equiv (1-x)^{n-1}$$

Upon equating the coefficients of  $x^i$  in the two members of this identity, we find

$$1 - A + B - C + \dots \pm L = (-1)^i \cdot \frac{(n-1)(n-2)(n-3) \dots (n-i)}{1^i} \\ = \left(1 - \frac{n}{1}\right) \left(1 - \frac{n}{2}\right) \left(1 - \frac{n}{3}\right) \dots \left(1 - \frac{n}{i}\right)$$

Now, the first member of this equation is the coefficient of  $f_0$  in (79); and since the final member contains only positive factors, it follows that the coefficient of  $f_0$  in (79) is a *positive* quantity. Accordingly, if we take  $f_0 = +0.5$ , in conjunction with the values of  $f_1, f_2, f_3, \dots$  designated in (80), the value of  $R$  given by (79) will then be the greatest possible under the assigned conditions.

We now append a table of the quantities  $f_0, f_1, f_2, f_3, \dots$  as above determined, with their differences :

$T$	$f$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$	$\Delta^{vii}$	(B')
$t$	<b>+0.5</b>	<b>0.0</b>							
$t + \omega$	+0.5	-1.0	-1	+3					
$t + 2\omega$	-0.5	+1.0	+2	-4	-7	+15			
$t + 3\omega$	+0.5	-1.0	-2	+4	+8	-16	-31	+63	
$t + 4\omega$	-0.5	+1.0	+2	-4	-8	+16	+32	-64	
$t + 5\omega$	+0.5		-2		+8		-32		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	

The special values which must be assigned to the quantities  $f_0, \alpha_0, \beta_0, \gamma_0, \dots$  of Table (B) are, therefore,

$$\begin{array}{ccccccccc} f_0 & \alpha_0 & \beta_0 & \gamma_0 & \delta_0 & \epsilon_0 & \dots & \dots & \dots \\ +0.5 & 0.0 & -1 & +3 & -7 & +15 & \dots & \dots & \dots \end{array}$$

in units of the last place of the tabular quantities  $F$ . Substituting these values in the original expression for  $R$  given in (77a), namely,

$$R = f_0 + A\alpha_0 + B\beta_0 + C\gamma_0 + \dots$$

we obtain

$$R = +0.5 - B + 3C - 7D + 15E - 31F + 63G - \dots \quad (81)$$

which gives the *maximum* value possible to  $R$  for  $n \geq 1$ .

To evaluate (81) for different values of  $n$  between 0 and  $+1$ , we make use of the following abridged table:

$n = A$	$B$	$C$	$D$	$E$	$F$	$G$
+	—	+	—	+	—	+
0.00	.0000	.0000	.0000	.0000	.0000	.0000
0.10	.0450	.0285	.0207	.0161	.0132	.0111
0.20	.0800	.0480	.0336	.0255	.0204	.0169
0.30	.1050	.0595	.0402	.0297	.0233	.0190
0.40	.1200	.0640	.0416	.0300	.0230	.0184
0.50	.1250	.0625	.0391	.0273	.0205	.0161
0.60	.1200	.0560	.0336	.0228	.0168	.0129
0.70	.1050	.0455	.0262	.0173	.0124	.0094
0.80	.0800	.0320	.0176	.0113	.0079	.0059
0.90	.0450	.0165	.0087	.0054	.0037	.0027
1.00	.0000	.0000	.0000	.0000	.0000	.0000
+	—	+	—	+	—	+

(D)

From these values we tabulate as follows :

$n$	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
		+	+	+	+	+	+	+	+	+	
— $B$	.000	.045	.080	.105	.120	.125	.120	.105	.080	.045	.000
+ $3C$	.000	.085	.144	.178	.192	.187	.168	.136	.096	.049	.000
— $7D$	.000	.145	.235	.281	.291	.274	.235	.183	.123	.061	.000
+ $15E$	.000	.241	.382	.445	.450	.409	.342	.259	.169	.081	.000
— $31F$	.000	.409	.632	.722	.713	.635	.521	.384	.245	.115	.000
+ $63G$	.000	.699	1.065	1.197	1.159	1.014	.813	.592	.372	.170	.000

If, now, we let  $R_2, R_3, R_4, \dots$  denote the values of  $R$  when differences beyond the 2d, 3d, 4th,  $\dots$  order respectively are neglected, then, from (81), we find

$$\left. \begin{aligned} R_2 &= 0.5 - B \\ R_3 &= 0.5 - B + 3C \\ R_4 &= 0.5 - B + 3C - 7D \\ &\dots \end{aligned} \right\}$$

(82)

From the last table we obtain, by successive additions, the values of  $R_2, R_3, R_4, \dots$  as defined by (82); these values are tabulated below :

$n$	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$R_2$	0.50	0.55	0.58	0.60	0.62	0.63	0.62	0.60	0.58	0.55	0.50
$R_3$	0.50	0.63	0.72	0.78	0.81	0.81	0.79	0.74	0.68	0.59	0.50
$R_4$	0.50	0.78	0.96	1.06	1.10	1.09	1.02	0.92	0.80	0.66	0.50
$R_5$	0.50	1.02	1.34	1.51	1.55	1.50	1.37	1.18	0.97	0.74	0.50
$R_6$	0.50	1.42	1.97	2.23	2.27	2.13	1.89	1.57	1.21	0.85	0.50
$R_7$	0.50	2.12	3.04	3.43	3.43	3.14	2.70	2.16	1.59	1.02	0.50



of Table (C) either are alternately  $+$  and  $-$ , or that  $+$  and  $-$  terms succeed each other irregularly. It follows that the quantities  $m$  must be numerically *less* than the *maximum* value of  $\mu$  in the series

$$\mu_0, \mu_1, \mu_2, \mu_3, \dots$$

For, otherwise, if the quantities  $m$  exceeded the greatest of the quantities  $\mu$ , the former would mask the effect of the latter in the combined series  $m + \mu$ ; hence there would be no general alternation of signs in the series

$$m_0 + \mu_0, \quad m_1 + \mu_1, \quad m_2 + \mu_2 \quad \dots$$

But this is contrary to our assumption that the differencing in Table (C) has been carried to an order  $\mathcal{A}^{(i+1)}$  which *does* exhibit a general alternation of signs. We therefore conclude that  $m_0$  is numerically less than the maximum value of  $\mu$ .

Now, from Table (B'), we observe that under the conditions assumed,

The maximum value of	$\alpha (= \mathcal{A}')$	is	$1 = (2)^0$ ;
"	"	"	" $\beta (= \mathcal{A}'')$ " $2 = (2)^1$ ;
"	"	"	" $\gamma (= \mathcal{A}''')$ " $4 = (2)^2$ ;
"	"	"	" $\mu (= \mathcal{A}^{(i+1)}) = (2)^i$ .

Hence,  $m_0$  is numerically less than  $2^i$ .

We have observed above that, as a consequence of the conditions herein assumed, the differences of  $\bar{F}$  in Table (A) are converging, being practically insensible beyond  $\mathcal{A}^{(i)}$ ; hence the fundamental expansions (0), and all relations deduced from these, are valid in this case. The formula (59) is therefore applicable to the series  $\bar{F}(T)$ ; hence, writing  $n$  for  $m$  in (59), we have

$$\delta_0' = Aa_0 + Bb_0 + Cc_0 + \dots + Ll_0 + Mm_0 + Nn_0 + \dots$$

in which as many terms should be retained as accuracy requires.

But we also have\*

$$\delta_0' = \bar{F}(t + n\omega) - \bar{F}(t) = \bar{F}_n - \bar{F}_0$$

and therefore

$$\bar{F}_n = \bar{F}_0 + Aa_0 + Bb_0 + Cc_0 + \dots + Ll_0 + Mm_0 + Nn_0 + \dots$$

---

\* See §26, where the same relations were similarly employed.

Now, by (84), this equation may be written

$$\bar{F}_n = Q + Mm_0 + Nn_0 + \dots$$

or

$$\bar{F}_n - Q = Mm_0 + Nn_0 + \dots \quad (85)$$

The series  $Mm_0 + Nn_0 + \dots$  therefore expresses the difference between the true mathematical value of the interpolated function and its approximate value  $Q$ . But since, as above observed, the differences  $m$  are nearly constant, it follows that the differences  $n$  are small in comparison. Hence,  $Nn_0$  is small as compared with  $Mm_0$ ; in brief,  $Mm_0$  represents, very nearly, the value of the rapidly converging series  $Mm_0 + Nn_0 + \dots$  in the right-hand member of (85). The latter equation may therefore be written, without sensible error,

$$\bar{F}_n - Q = Mm_0 \quad (86)$$

From (82) we derive

$$\left. \begin{aligned} R_3 - R_2 &= +3C = (2^2 - 1)(+C) \\ R_4 - R_3 &= -7D = (2^3 - 1)(-D) \\ R_5 - R_4 &= +15E = (2^4 - 1)(+E) \\ &\dots \dots \dots \\ R_{i+1} - R_i &= (2^i - 1)(-1)^i M \end{aligned} \right\} \quad (87)$$

From the last of these, we obtain

$$\pm 2^i M = R_{i+1} - R_i \pm M \quad (88)$$

We have shown above that  $m_0$  is numerically less than  $2^i$ ; this condition may be expressed in the form

$$m_0 = 2^i \sin \theta$$

where  $\theta$  may have *any* value between 0 and  $2\pi$ . From this relation we obtain

$$Mm_0 = 2^i M \sin \theta$$

or, by (88),

$$Mm_0 = (R_{i+1} - R_i \pm M) \sin \theta \quad (89)$$

Substituting this value of  $Mm_0$  in (86), we get

$$\bar{F}_n - Q = (R_{i+1} - R_i \pm M) \sin \theta \quad (90)$$

From (77a), we have\*

$$F_n - Q = R_i \quad (91)$$

which, subtracted from (90), gives

$$\bar{F}_n - F_n = R_{i+1} \sin \theta - (1 + \sin \theta) R_i \pm M \sin \theta$$

From Table (D) above we see that beyond  $\mathcal{A}'''$  the coefficient  $M$  cannot exceed 0.04, which is an inappreciable quantity in the present discussion; we therefore write the last equation

$$\bar{F}_n - F_n = R_{i+1} \sin \theta - (1 + \sin \theta) R_i \quad (92)$$

The quantity  $R_{i+1}$  is numerically greater than  $R_i$ , and both are alike in sign; this condition may be expressed by the relation

$$R_i = R_{i+1} \sin^2 \psi$$

in which  $\psi$  has a definite value depending upon the value of  $i$ . Substituting this expression for  $R_i$  in (92), the latter becomes

$$\bar{F}_n - F_n = R_{i+1} [\sin \theta - \sin^2 \psi (1 + \sin \theta)]$$

or

$$\bar{F}_n - F_n = R_{i+1} (\sin \theta \cos^2 \psi - \sin^2 \psi) \quad (93)$$

Since  $\cos^2 \psi$  is necessarily positive, and  $-\sin^2 \psi$  negative, it follows that the coefficient of  $R_{i+1}$  in (93) will be numerically a maximum when  $\sin \theta$  attains its greatest negative value; that is, when  $\theta = \frac{3}{2} \pi$ . Taking  $\theta = \frac{3}{2} \pi$  in (93), we have

$$\bar{F}_n - F_n = R_{i+1} (-\cos^2 \psi - \sin^2 \psi) = -R_{i+1} \quad (94)$$

which is the maximum numerical value possible to  $\bar{F}_n - F_n$ , all conditions favoring.

$\bar{F}_n$  is the *true mathematical value* of the required function.  $F_n$  is the approximate value of this quantity which is obtained by applying NEWTON'S Formula to Table (C), neglecting differences beyond  $\mathcal{A}^{(i)}$ : it being assumed, (1) that the given functions  $F_0, F_1, F_2, F_3, \dots$  are true to the *nearest* unit of their last digit; (2) that  $n$  is positive

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\* The quantity  $R$  defined in (77a) is not distinguished by a subscript in the earlier part of this discussion. Considered as a particular term of the series  $R_2, R_3, R_4, \dots$ , however, it is evident that  $R$  should be designated as  $R_i$ .

and less than unity; (3) that the differences  $\Delta^{(i)}$  are approximately constant; and (4) that the differences  $\Delta^{(i+1)}$  are quite small, with + and — signs following irregularly. Under these conditions, it follows from (94) that the computed value  $F_n$  can never differ from the true value  $\bar{F}_n$  by more than the quantity  $R_{i+1}$ .

One point further, however, must be considered. In computing  $F_n$  by (76), we should, in practice, obtain the values of the several terms to one or two decimals further than are given in  $F$ , to avoid accumulation of errors in the final addition. But in writing the sum,  $F_n$ , the extra decimals are dropped, the result being taken to the *nearest unit*, as in  $F$ . Thus we actually use, not the quantity  $F_n$  obtained rigorously by (76), but a close approximation to that value, which we may denote by  $(F_n)$ . Accordingly, the relation

$$F_n - (F_n) = \pm 0.5$$

expresses the maximum discrepancy between  $F_n$  and  $(F_n)$ . Combining this expression with (94), we finally obtain

$$\bar{F}_n - (F_n) = -R_{i+1} \pm 0.5 \tag{95}$$

The quantity  $R_{i+1} \pm 0.5$  therefore represents the final limit of error in the value of an interpolated function, in units of the last decimal of  $F$ . From the value of  $R_6$  given in (83), we find that when  $\Delta^r$  is nearly constant, the limiting error is  $\pm 2.8$  units. Since it is highly improbable that all the necessary conditions will conspire to produce this *maximum* error, we may add that when the differences practically terminate at the fifth order, interpolated functions will *occasionally* be in error by one unit, *only rarely* in error by two units, and *never* by three.

With sixth, seventh, or higher differences employed, the results become subject to errors which in most cases would be intolerable, and which would probably be obviated by a direct calculation of the function.

From the foregoing investigation it therefore appears that, for purposes of interpolation, tabular functions should always be given with an interval sufficiently small that differences beyond  $\Delta^r$  may be

neglected. This condition is generally fulfilled in practice. As already stated in §24, the longitude and latitude of the moon are given in the *Nautical Almanac* for every twelve hours; from the values thus given, intermediate positions can always be safely interpolated by using differences no higher than the fourth or fifth order. On the other hand, a table of the moon's longitude for every 24 hours would yield differences of the eighth or even ninth order; the use of which in NEWTON'S Formula might produce an error of several units in an interpolated position.

In all that follows, we shall assume that differences beyond the fifth order may be neglected. This assumption made, it follows from the preceding investigation that the fundamental formulae, (73) and (75), may be applied in all cases without sensible error, provided that  $n$  is taken less than unity.

31. We shall now solve an example which illustrates the main points of the foregoing discussion. If we tabulate the function

$$\bar{F}(T) \equiv \tau \frac{1}{21600} \left\{ \begin{array}{rcl} 606607.920 & -199841.772\,T + 50804.968\,T^2 & \\ + 5645.715\,T^3 - 2169.395\,T^4 + 116.817\,T^5 + 1.507\,T^6 & \end{array} \right\} \quad (96)$$

for  $T = 0, 1, 2, 3, \dots, 9$ , we find that the true mathematical values terminate in the fifth decimal. These values of  $\bar{F}(T)$  are given in the table below, with their differences:

$T$	$\bar{F}(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$
0	8.42511						
1	6.40508	-2.02003					
2	5.89492	-0.51016	+1.50987				
3	6.53508	+0.64016	1.15032	-0.35955			
4	7.66492	1.12984	+0.48968	0.66064	-0.30109	+ .23237	+ .01507
5	8.55508	0.89016	-0.23968	0.72936	-0.06872	.24744	.01507
6	8.65492	+0.09984	0.79032	0.55064	+0.17872	.26251	.01507
7	7.85503	-0.79989	0.89973	-0.10941	0.44123	.27758	+ .01507
8	6.76481	-1.09022	-0.29033	+0.60940	0.71881	+ .29265	
9	7.00512	+0.24031	+1.33053	+1.62086	+1.01146		

This table corresponds to Table (A) of the last section. It will be observed that the values of  $\bar{F}$  are peculiar from the fact that the

last three decimals of each differ *only slightly* from the quantity 0.00500, or half a unit in the second decimal place; and, moreover, that the actual difference is, excepting the first function, *alternately in excess and defect*. This condition will rarely obtain, and is here selected only to illustrate the limiting case.

If now we drop the last three decimals of  $\overline{F}$ , we obtain a series of approximate values, denoted by  $F$ . The following table gives  $F$ , true to the nearest unit of the second decimal, together with its differences:

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$	
0	8.43	-2.02						
1	6.41	-0.52	+1.50					
2	5.89	+0.65	1.17	-0.33	-0.37	+0.38		
3	6.54	1.12	+0.47	0.70	+0.01	0.09	-0.29	(C')
4	7.66	0.90	-0.22	0.69	0.10	0.42	+0.33	
5	8.56	+0.09	0.81	0.59	0.52	0.12	-0.30	
6	8.65	-0.79	0.88	-0.07	0.64	0.12	+0.33	
7	7.86	-1.10	-0.31	+0.57	+1.09	+0.45		
8	6.76	+0.25	+1.35	+1.66				
9	7.01							

Table (C') corresponds to Table (C) of §30. It will be observed that  $\Delta^v$  and  $\Delta^{vi}$ , in (C'), represent  $\Delta^{(v)}$  and  $\Delta^{(v+1)}$ , of Table (C). The differencing in (C') is not carried beyond  $\Delta^{vi}$ , because of the alternation of  $+$  and  $-$  terms.

The above values of  $F$  may be written as follows:

$$\begin{aligned} F &= \overline{F} + f \\ 8.43 &= 8.42511 + 0.00489 \\ 6.41 &= 6.40508 + 0.00492 \\ 5.89 &= 5.89492 - 0.00492 \\ &\dots\dots\dots \end{aligned}$$

The quantities in the last column therefore represent the residual terms denoted by  $f$  in the preceding section. Expressing these values in units of the second decimal, we have the following table of  $f$  and its differences:

$T$	$f$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$	$\Delta^{vi}$
0	+0.489	+0.003					
1	+0.492	−0.984	−0.987	+2.955			
2	−0.492	+0.984	+1.968	−3.936	−6.891	+14.763	
3	+0.492	−0.984	−1.968	+3.936	+7.872	−15.744	−30.507
4	−0.492	+0.984	+1.968	−3.936	−7.872	+15.749	+31.493
5	+0.492	−0.984	−1.968	+3.941	+7.877	−15.758	−31.507
6	−0.492	+0.989	+1.973	−3.940	−7.881	+15.735	+31.493
7	+0.497	−0.978	−1.967	+3.914	+7.854		
8	−0.481	+0.969	+1.947				
9	+0.488						

(B'')

It will be observed that the quantities of Table (B'') are close approximations to the (limiting) values given in Table (B'), of §30.

Let us now apply NEWTON'S Formula to interpolate the value of  $F$  which corresponds to  $T=0.40$ , in Table (C'). Neglecting differences beyond  $\Delta^v$ , we take from Table I (for  $n=0.40$ ), and from Table (C'), the quantities to be employed. The result is as follows:

		$F_0 = +8.43$
$A = +0.40$	$a = -2.02$	$Aa = -0.8080$
$B = -0.12$	$b = +1.50$	$Bb = -0.1800$
$C = +0.064$	$c = -0.33$	$Cc = -0.0211$
$D = -0.0416$	$d = -0.37$	$Dd = +0.0154$
$E = +0.02995$	$e = +0.38$	$Ee = +0.0114$
		$\therefore F_n = +7.4477$

Whence, we write for the value of the interpolated function,

$$\left. \begin{aligned} (F_n) &= 7.45 \\ &= 7.44,77 + 0.00,23 = F_n + 0.00,23 \end{aligned} \right\} \tag{97}$$

Computing the true value  $\overline{F}_n$  from (96), we obtain

$$\overline{F}_n = 7.4320416 + \tag{98}$$

Hence the value  $(F_n)=7.45$ , interpolated from Table (C'), is in error by 1.8 units of its last place.

The value of  $Q$  is the result obtained by interpolating  $\overline{F}_n$  from Table (A'), neglecting differences after  $\Delta^v$ . Thus we determine  $Q$  as follows:

		$\bar{F}_0 = +8.425110$
$A = +0.40$	$a_0 = -2.02003$	$Aa_0 = -0.808012$
$B = -0.12$	$b_0 = +1.50987$	$Bb_0 = -0.181184 +$
$C = +0.064$	$c_0 = -0.35955$	$Cc_0 = -0.023011 +$
$D = -0.0416$	$d_0 = -0.30109$	$Dd_0 = +0.012525 +$
$E = +0.02995$	$e_0 = +0.23237$	$Ee_0 = +0.006959 +$
		$\therefore Q = +7.432387 +$

The value of  $R_5$  is computed from Table (B'') in the same manner that  $Q$  has just been obtained from (A'). Thus we find

		$f_0 = +0.489$
$A = +0.40$	$\alpha_0 = + 0.003$	$A\alpha_0 = +0.001$
$B = -0.12$	$\beta_0 = - 0.987$	$B\beta_0 = +0.118$
$C = +0.064$	$\gamma_0 = + 2.955$	$C\gamma_0 = +0.189$
$D = -0.0416$	$\delta_0 = - 6.891$	$D\delta_0 = +0.287$
$E = +0.02995$	$\epsilon_0 = +14.763$	$E\epsilon_0 = +0.442$
$\therefore$ (In units of the second decimal)		$R_5 = +1.526$ [Cf. (83)]

Now, from (91) we have

$$F_n = Q + R_5 \tag{99}$$

Substituting the above values of  $Q$  and  $R_5$ , we find

$$F_n = 7.4324 + 0.01,53 = 7.4477$$

which agrees with the result obtained directly from Table (C').

Since the sixth differences in Table (A') are constant, it follows that the true value  $\bar{F}_n$  differs from the above value of  $Q$  only by the term in  $\Delta^vi$  of NEWTON'S Formula. Now, the coefficient of  $\Delta^vi$  is found from Table (D) of the last section to be approximately  $-0.0230$ . Hence, with  $\Delta^vi = +0.01507$ , we derive

$$\left. \begin{aligned} \bar{F}_n &= Q - (0.0230 \times 0.01507) \\ &= Q - 0.000346 \\ &= 7.432387 - 0.000346 \\ &= 7.432041 \end{aligned} \right\} \text{ (nearly)}$$

which agrees with (98). The second of these equations gives

$$Q = \bar{F}_n + 0.000346 \pm$$

Substituting this value of  $Q$  in (99), we have

$$F_n = \bar{F}_n + R_5 + 0.0346$$

where the numerical term is now expressed in the same unit as  $R_5$ . With the above determined value of  $R_5(=+1.526)$ , the last equation becomes

$$F_n = \bar{F}_n + 1.56$$

Finally, since we were obliged to write  $(F_n)$  greater than  $F_n$  by 0.23 units, it follows that the actual error of interpolation in this instance is  $1.56+0.23$ , or approximately 1.8 units in the second decimal place; which agrees with the result previously obtained.

32. As a more practical application of NEWTON'S Formula, we take the following

EXAMPLE.—From the appended table, find the sun's right-ascension for April 20<sup>d</sup> 0<sup>h</sup>.

Date 1898	Sun's R. A.	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
	<sup>h</sup> <sup>m</sup> <sup>s</sup>	<sup>m</sup> <sup>s</sup>	<sup>s</sup>	<sup>s</sup>	<sup>s</sup>
April 1	0 43 20.30				
6	1 1 34.07	+18 13.77	+ 5.15	+2.53	
11	1 19 52.99	18 18.92	7.68	1.97	−0.56
16	1 38 19.59	18 26.60	9.65	1.34	0.63
21	1 56 55.84	18 36.25	10.99	1.05	−0.29
26	2 15 43.08	18 47.24	12.04	+1.06	+0.01
May 1	2 34 42.36	18 59.28			
6	2 53 54.74	+19 12.38	+13.10		

Letting  $t = \text{April 16}$ , we have

$$n = \frac{20-16}{5} = 0.80$$

Then, from Table I, and the above differences, we find

		<sup>m</sup> <sup>s</sup>	$F_0 = 1^{\text{h}} 38^{\text{m}} 19.59^{\text{s}}$
$A = +0.80$	$a_0 = +18$	36.25	$Aa_0 = +0 14 53.000$
$B = -0.08$	$b_0 = +$	10.99	$Bb_0 = - 0.879$
$C = +0.032$	$c_0 = +$	1.05	$Cc_0 = + 0.034$
$D = -0.0176$	$d_0 = +$	0.01	$Dd_0 = 0.000$
$\therefore \text{Sun's R.A., April 20}^{\text{d}} 0^{\text{h}}$			$= 1 53 11.75$

which is the value given in the *American Ephemeris* for 1898.

33. Since the value of  $n$  in the preceding example is only 0.2 less than unity, it is more convenient to interpolate *backwards* from

April 21, by means of (75). Thus, from Table I (for  $n = 0.20$ ), and the tabular differences, we find

			$F_0$	$=$	$1^{\text{h}} 56^{\text{m}} 55.84^{\text{s}}$
$A = +0.20$	$a_{-1} = +18$	$36.25^{\text{s}}$	$-Aa_{-1}$	$=$	$-0^{\text{h}} 3^{\text{m}} 43.250^{\text{s}}$
$B = -0.08$	$b_{-2} = +$	$9.65$	$+Bb_{-2}$	$=$	$0.772$
$C = +0.048$	$c_{-3} = +$	$1.97$	$-Cc_{-3}$	$=$	$0.095$
$D = -0.0336$	$d_{-4} = -$	$0.56$	$+Dd_{-4}$	$=$	$0.019$
$\therefore$ Sun's R.A., April 20 <sup>d</sup> 0 <sup>h</sup>				$=$	$1^{\text{h}} 53^{\text{m}} 11.74^{\text{s}}$

which agrees within 0<sup>s</sup>.01 of the first result. Whenever a check is considered necessary, the interpolation may be performed by both methods.

TRANSFORMATIONS OF NEWTON'S FORMULA.

34. *Modification of the Foregoing Notation of Differences: STIRLING'S Formula.*—In NEWTON'S Formula of interpolation we use differences which depend only upon the functions  $F_0, F_1, F_2, \dots$ ; the functions preceding  $F_0$ , whether given or not, are in no way involved. We shall now transform NEWTON'S Formula in such a manner as to involve differences both preceding and following the function from which we set out. The resulting formulae will in general be more convenient, rapidly convergent, and accurate than NEWTON'S Formula.

In the schedule below, the preceding notation of differences is modified: the *even* differences which fall on the horizontal line through  $F_0$  are now denoted by the subscript *zero*, as  $b_0$  and  $d_0$ ; all differences *above* this line are indicated by *accents*, as  $a', b', c,$ " etc.; while all differences *below* the horizontal line through  $F_0$  are indicated by subscripts, as  $a_1, b_1, c_2$ , etc. The new schedule of differences will then be as follows:

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{\text{iv}}$	$\Delta^{\text{v}}$
$t - 2\omega$	$F_{-2}$					
$t - \omega$	$F_{-1}$	$a''$	$b'$	$c''$	$d'$	$e''$
$t$	$F_0$	$a'$	$b_0$	$c'$	$d_0$	$e'$
$t + \omega$	$F_1$	$a_1$	$b_1$	$c_1$	$d_1$	$e_1$
$t + 2\omega$	$F_2$	$a_2$	$b_2$	$c_2$	$d_2$	$e_2$
$t + 3\omega$	$F_3$	$a_3$		$c_3$		$e_3$

*To derive STIRLING'S Formula:* Applying NEWTON'S Formula to the above schedule, we find for the value of  $F_n$ ,

$$F_n = F_0 + na_1 + Bb_1 + Cc_2 + Dd_2 + Ee_3 + \dots \quad (100)$$

where, as before,  $B, C, D, E, \dots$  represent the binomial coefficients of  $\Delta'', \Delta''', \Delta^{IV}, \Delta^V, \dots$ , respectively. Let us now put

$$a = \frac{1}{2}(a' + a_1) \quad , \quad c = \frac{1}{2}(c' + c_1) \quad , \quad e = \frac{1}{2}(e' + e_1) \quad (101)$$

from which, with the relations

$$a_1 - a' = b_0 \quad , \quad c_1 - c' = d_0 \quad , \quad e_1 = e' + \dots$$

we obtain

$$a_1 = a + \frac{1}{2}b_0 \quad , \quad c' = c - \frac{1}{2}d_0 \quad , \quad c_1 = c + \frac{1}{2}d_0 \quad , \quad e_1 = e + \dots \quad (102)$$

Using the equations (102), together with the relations given in §23, we find

$$\left. \begin{aligned} a_1 &= a + \frac{1}{2}b_0 \\ b_1 &= b_0 + c_1 = b_0 + c + \frac{1}{2}d_0 \\ c_2 &= c' + 2d_0 + e_1 = c + \frac{3}{2}d_0 + e + \dots \\ d_2 &= d_0 + 2e_1 + \dots = d_0 + 2e + \dots \\ e_3 &= e_1 + \dots = e + \dots \end{aligned} \right\} \quad (103)$$

Upon substituting these values of  $a_1, b_1, c_2, \dots$  in (100), the latter becomes

$$\begin{aligned} F_n &= F_0 + n(a + \frac{1}{2}b_0) + B(b_0 + c + \frac{1}{2}d_0) + C(c + \frac{3}{2}d_0 + e + \dots) + D(d_0 + 2e + \dots) + Ee + \dots \\ &= F_0 + na + (B + \frac{n}{2})b_0 + (C + B)c + (D + \frac{3}{2}C + \frac{1}{2}B)d_0 + (E + 2D + C)e + \dots \end{aligned}$$

Substituting in the last equation the values of  $B, C, D, E$ , namely,

$$\begin{aligned} B &= \frac{n(n-1)}{1^2} \quad , \quad D = \frac{n(n-1) \dots (n-3)}{1^4} \\ C &= \frac{n(n-1)(n-2)}{1^3} \quad , \quad E = \frac{n(n-1) \dots (n-4)}{1^5} \end{aligned}$$

we finally obtain

$$F_n = F_0 + na + \frac{n^2}{2}b_0 + \frac{n(n^2-1)}{6}c + \frac{n^2(n^2-1)}{24}d_0 + \frac{n(n^2-1)(n^2-4)}{120}e + \dots \quad (104)$$

which is known as STIRLING'S Formula. The even differences employed in this formula are those falling on the horizontal line through

$F_0$ ; the odd differences are the *means* of those which fall immediately above and below this line, as defined by (101).

Table II gives the values of STIRLING'S coefficients for the argument  $n$ . A glance at this table shows how much more rapidly these coefficients converge than those of NEWTON'S Formula.

EXAMPLE.—From the table below, find the R.A. of the sun for April 20<sup>d</sup> 0<sup>h</sup>.

Date 1898	Sun's R.A.	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
	<sup>h</sup> <sup>m</sup> <sup>s</sup>	<sup>m</sup> <sup>s</sup>	<sup>s</sup>	<sup>s</sup>	<sup>s</sup>
April 1	0 43 20.30	+18 13.77			
6	1 1 34.07	18 18.92	+ 5.15	+2.53	
11	1 19 52.99	18 26.60	7.68	1.97	−0.56
16	1 38 19.59		9.65		0.63
21	1 56 55.84	18 36.25	10.99	1.34	−0.29
26	2 15 43.08	18 47.24	12.04	1.05	+0.01
May 1	2 34 42.36	18 59.28	+13.10	+1.06	
6	2 53 54.74	+19 12.38			

Taking  $t = \text{April 16}$  (as in §32), we have

$$n = \frac{20-16}{5} = 0.80$$

The horizontal lines drawn in the body of the table indicate the differences to be employed in (104), as follows:

- (1) The required values of  $F_0$ ,  $\Delta''$ , and  $\Delta^{iv}$  are those *included between* two lines;
- (2) The required values of  $\Delta'$  and  $\Delta'''$  are the *means* of the quantities *separated by* a single line.

As before, we shall denote the coefficients of  $\Delta'$ ,  $\Delta''$ ,  $\Delta'''$ , . . . by  $A$ ,  $B$ ,  $C$ , . . . . Taking their values from Table II, with  $n = 0.80$ , and forming the required differences as indicated, we obtain

		<sup>m</sup> <sup>s</sup>	$F_0 =$	1 <sup>h</sup> 38 <sup>m</sup> 19.59 <sup>s</sup>
$A = +0.80$	$a = +18$	31.425	$Aa = +$	14 49.140
$B = +0.32$	$b_0 = +$	9.65	$Bb_0 = +$	3.088
$C = -0.048$	$c = +$	1.66	$Cc = -$	0.080
$D = -0.0096$	$d_0 = -$	0.63	$Dd_0 = +$	0.006
$\therefore \text{Sun's R.A., April 20}^d \text{ 0}^h$			$=$ 1 53 11.74	

which agrees exactly with the result found in §33.

35. *Backward Interpolation by STIRLING'S Formula.*—When the *forward* interval approaches unity, it will be more convenient to proceed *backwards* from the following function by the formula

$$F_{-n}= F_0 - na + \frac{n^2}{2} b_0 - \frac{n(n^2-1)}{6} c + \frac{n^2(n^2-1)}{24} d_0 - \frac{n(n^2-1)(n^2-4)}{120} e + \dots \tag{105}$$

the coefficients of which are taken from Table II with the argument  $n$ , as before. It will be observed that (105) is derived from (104) by merely writing  $-n$  for  $n$  in the latter; or, by supposing the given series to be inverted, and hence (Theorem III) changing the signs of  $a$ ,  $c$ , and  $e$ .

EXAMPLE.—Solve the preceding example by (105); that is, find the sun's R.A. for April 20<sup>d</sup> 0<sup>h</sup> by backward interpolation.

Taking  $t =$  April 21, we have

$$n = \frac{21-20}{5} = 0.20$$

The differences are formed for the date April 21 in the same manner as found above for April 20; thence, taking the coefficients from Table II, with  $n = 0.20$ , we find

			$F_0 =$	1 <sup>h</sup> 56 <sup>m</sup> 55.84 <sup>s</sup>
$A = +0.20$	$a = +18$	41.745 <sup>m s</sup>	$-Aa = -$	3 44.349
$B = +0.02$	$b_0 = +$	10.99	$+Bb_0 = +$	0.220
$C = -0.032$	$c = +$	1.20	$-Cc = +$	0.038
$D = -0.0016$	$d_0 = -$	0.29	$+Dd_0 =$	0.000
$\therefore$ Sun's R.A., April 20 <sup>d</sup> 0 <sup>h</sup>				$=$ 1 53 11.75

36. EXAMPLE.—Use STIRLING'S Formula to compute  $\log \sin 9^\circ 22'$  from the following table:

$T$	$\text{Log sin } T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{\text{iv}}$	$\Delta^{\text{v}}$
6 <sup>°</sup>	9.01923	+6666				
7	9.08589	5767	−899			
8	9.14356	5077	690	+209		
9	9.19433		543	147	−62	+17
10	9.23967	4534	441	102	45	
11	9.28060	4093	−365	+ 76	−26	+19
12	9.31788	+3728				

Here we have

$$t = 9^\circ \qquad n = \frac{22}{60} = 0.36667$$

and we therefore obtain

		$F_0 = 9.19433$
$A = +0.36667$	$a = +4805.5$	$Aa = + 1762.0$
$B = +0.06722$	$b_0 = - 543$	$Bb_0 = - 36.5$
$C = -0.05289$	$c = + 124.5$	$Cc = - 6.6$
$D = -0.00485$	$d_0 = - 45$	$Dd_0 = + 0.2$
$E = +0.01022$	$e = + 18$	$Ee = + 0.2$
$\therefore \text{Log sin } 9^\circ 22'$		$= 9.21152.3$

The true value to six decimals is 9.211526.

37. *The Algebraic Mean.*—It may be well to observe that in taking the mean of two quantities having like signs, and of nearly the same magnitude, it is easier to add one-half their *difference* to the lesser number, than to take one-half the sum of the two quantities. That is, we proceed according to the identity

$$\frac{1}{2}(x+y) = x + \frac{1}{2}(y-x)$$

in which we suppose  $y$  numerically greater than  $x$ . Thus, in the last example, instead of taking

$$a = \frac{1}{2}(a' + a_1) = \frac{1}{2}(5077 + 4534) = \frac{1}{2}(9611) = +4805.5$$

it is easier to follow the equivalent formula

$$a = a_1 - \frac{1}{2}(a_1 - a') = a_1 - \frac{1}{2}b_0 = 4534 + \frac{1}{2}(543) = +4805.5$$

Similarly, we find

$$c = 102 + 22.5 = +124.5$$

*Per contra*, to form the mean of two quantities having unlike signs, and differing but little in magnitude, it is easier to take their algebraic sum and then divide by two. For example, given the values

$F(T)$	$\Delta'$	$\Delta''$
$F_{-1}$	-4226 +5088	+9314
$F_0$		
$F_1$		

we find

$$a = \frac{1}{2}(5088 - 4226) = \frac{1}{2}(+862) = +431$$

With these precepts, the required *mean differences* of interpolation are very readily taken.

38. *BESSEL'S Formula.*—We now pass from *STIRLING'S Formula* to another, somewhat similar, wherein we employ the odd differences  $a_1, c_1, e_1$ , which fall on the horizontal line between  $F_0$  and  $F_1$ , and the *means* of the even differences falling immediately above and below this line. Using the schedule on page 62, let us put

$$b = \frac{1}{2}(b_0 + b_1) \quad , \quad d = \frac{1}{2}(d_0 + d_1) \quad (106)$$

Then, since  $b_1 - b_0 = c_1$ , and  $d_1 - d_0 = e_1$ , these equations give

$$b_0 = b - \frac{1}{2}c_1 \quad , \quad d_0 = d - \frac{1}{2}e_1 \quad (107)$$

Let us write the formula (104), for brevity,

$$F_n = F_0 + na + \frac{n^2}{2}b_0 + Cc + Dd_0 + Ee + \dots \quad (108)$$

where

$$C = \frac{n(n^2-1)}{6} \quad , \quad D = \frac{n^2(n^2-1)}{24} \quad , \quad E = \frac{n(n^2-1)(n^2-4)}{120} \quad (109)$$

Now, by means of (102) and (107), we derive

$$\left. \begin{aligned} a &= a_1 - \frac{1}{2}b_0 = a_1 - \frac{1}{2}(b - \frac{1}{2}c_1) = a_1 - \frac{1}{2}b + \frac{1}{4}c_1 \\ b_0 &= b - \frac{1}{2}c_1 \\ c &= c_1 - \frac{1}{2}d_0 = c_1 - \frac{1}{2}(d - \frac{1}{2}e_1) = c_1 - \frac{1}{2}d + \frac{1}{4}e_1 \\ d_0 &= d - \frac{1}{2}e_1 \\ e &= e_1 - \dots \end{aligned} \right\} \quad (110)$$

Upon substituting these values of  $a, b_0, c, \dots$  in (108), we have

$$\begin{aligned} F_n &= F_0 + n(a_1 - \frac{1}{2}b + \frac{1}{4}c_1) + \frac{n^2}{2}(b - \frac{1}{2}c_1) + C(c_1 - \frac{1}{2}d + \frac{1}{4}e_1) + D(d - \frac{1}{2}e_1) + E(e_1 - \dots) + \dots \\ &= F_0 + na_1 + (\frac{n^2}{2} - \frac{n}{2})b + (C - \frac{n^2}{4} + \frac{n}{4})c_1 + (D - \frac{1}{2}C)d + (E - \frac{1}{2}D + \frac{1}{4}C)e_1 + \dots \end{aligned}$$

Finally, substituting in the last equation the values of  $C, D, E$ , from (109), we obtain

$$\begin{aligned} F_n &= F_0 + na_1 + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-\frac{1}{2})}{6}c_1 \\ &\quad + \frac{(n+1)n(n-1)(n-2)}{24}d + \frac{(n+1)n(n-1)(n-2)(n-\frac{1}{2})}{120}e_1 + \dots \quad (111) \end{aligned}$$

which is *BESSEL'S Formula* of interpolation, commonly regarded as the most convenient and accurate of the several forms in use. The odd differences here employed are those which fall on the horizontal line between  $F_0$  and  $F_1$ , as shown in the schedule on page 62; the even differences are the *means* of those falling immediately above and below this line, as defined by (106).

Table III gives BESSEL'S coefficients for the argument  $n$ .

EXAMPLE.—Use BESSEL'S Formula to compute  $\log \sin 9^\circ 22'$  from the table below:

$T$	$\text{Log sin } T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$6^\circ$	9.01923					
7	9.08589	+6666	—899			
8	9.14356	5767	690	+209	—62	
		5077	543	147	45	+17
9	9.19433	4534	441	102	—26	+19
		4093	—365	+76		
10	9.23967	+3728				
11	9.28060					
12	9.31788					

We have, as in §36,

$t = 9^\circ$

$n = 0.36667$

The horizontal lines drawn in the table indicate that the values of  $F_0$ ,  $\Delta'$ ,  $\Delta'''$  and  $\Delta^v$ , to be employed in (111), are those included between the parallel lines; while the required values of  $\Delta''$  and  $\Delta^{iv}$  are the *means* of the quantities separated by the single line. Forming the differences thus indicated and taking their coefficients from Table III, with  $n = 0.36\frac{2}{3}$ , we obtain

$A = +0.36667$

$a_1 = +4534$

$F_0 = 9.19433$

$B = -0.11611$

$b = -492$

$Aa_1 = +1662.5$

$C = +0.00516$

$c_1 = +102$

$Bb = +57.1$

$D = +0.02160$

$d = -36$

$Cc_1 = +0.5$

$E = -0.00057$

$e_1 = +19$

$Dd = -0.8$

$Ee_1 = 0.0$

$\therefore \text{Log sin } 9^\circ 22' = 9.21152.3$

which agrees exactly with the value found in §36.

39. EXAMPLE.—Find by BESSEL'S Formula the value of  $10^4$  from the following table of  $T^4$ .

$T$	$T^4$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
— 8	+ 4096	— 4015				
— 3	81	— 65	+ 3950	— 1500		
+ 2	16	+ 2385	2450	+13500	+15000	0
			15950		15000	
7	2401	18335	44450	28500		0
		62785	+87950	+43500	+15000	
12	20736	+150735				
17	83521					
+22	+234256					

Taking  $t = 7$ , we have

$$n = \frac{10-7}{5} = 0.60$$

Therefore we find

		$F_0 = + 2401$
$A = +0.60$	$a_1 = +18335$	$Aa_1 = +11001$
$B = -0.120$	$b = +30200$	$Bb = - 3624$
$C = -0.0040$	$c_1 = +28500$	$Cc_1 = - 114$
$D = +0.0224$	$d = +15000$	$Dd = + 336$
		$\therefore 10^4 = +10000$

40. *Backward Interpolation by BESSEL'S Formula.*—To find  $F_{-n}$  by BESSEL'S Formula, we conceive the series given on page 62 to be inverted; the required function is then found by interpolating *foward* from  $F_0$  toward  $F_{-1}$  with the interval  $n$ . Hence, the differences to be used in (111) are—

$$-a', \quad +\frac{1}{2}(b_0+b'), \quad -c', \quad +\frac{1}{2}(d_0+d'), \quad -e', \quad \dots$$

We therefore have

$$F_{-n} = F_0 - na' + \frac{n(n-1)}{2} \cdot \frac{b_0+b'}{2} - \frac{n(n-1)(n-\frac{1}{2})}{6} c' + \dots \quad (111a)$$

the coefficients, as in (111), being taken from Table III with the argument  $n$ .

EXAMPLE.—Find  $10^4$  from the table of §39, by means of (111a).

Taking  $t = 12$ , we find

$$n = \frac{12-10}{5} = 0.40$$

The differences are here the same as in the last example; thus we obtain

		$F_0 = +20736$
$A = +0.40$	$a' = +18335$	$-Aa' = - 7334$
$B = -0.120$	$\frac{b_0+b'}{2} = +30200$	$+B \cdot \frac{b_0+b'}{2} = - 3624$
$C = +0.0040$	$c' = +28500$	$-Cc' = - 114$
$D = +0.0224$	$\frac{d_0+d'}{2} = +15000$	$+D \cdot \frac{d_0+d'}{2} = + 336$
		$\therefore 10^4 = +10000$

41. *Property of BESSEL'S Coefficients.*—If we take from Table III the coefficients for  $A'', A''', A^{iv}, A^v$ , with the argument  $n = 0.30$ , and also with  $n = 0.70$  ( $= 1.00 - 0.30$ ), we find the following values:

$n$	$B$	$C$	$D$	$E$
0.30	-.10500	+.00700	+.01934	-.00077
0.70	-.10500	-.00700	+.01934	+.00077

It will be observed that the coefficients are here numerically the same for the arguments  $n$  and  $1-n$ ; having like signs for the even orders, and opposite signs for the odd orders of differences.

More generally, let us denote the values of BESSEL'S coefficients for  $\Delta''$ ,  $\Delta'''$ ,  $\Delta^{\text{iv}}$ ,  $\Delta^{\text{v}}$ , . . . . taken with the argument  $n$ , by  $B, C, D, E, \dots$ , respectively; and the corresponding values taken with the argument  $1-n$  by  $B_1, C_1, D_1, E_1, \dots$ . An inspection of Table III then shows that we have

$$\left. \begin{aligned} B_1 &= +B \\ C_1 &= -C \\ D_1 &= +D \\ E_1 &= -E \\ &\dots \end{aligned} \right\} \quad (112)$$

To establish these relations generally, we write (111) in the form

$$F_n = F_0 + na_1 + Bb + Cc_1 + Dd + Ee_1 + \dots \quad (113)$$

Now, the value of  $F_n$  may also be obtained by interpolating *backwards* from  $F_1$  with the interval  $1-n$ ; the differences thus involved will be exactly the same as in (113). Hence, after the manner of formula (111a), we have

$$F_n = F_1 - (1-n)a_1 + B_1b - C_1c_1 + D_1d - E_1e_1 + \dots \quad (114)$$

But we have, also,

$$F_1 - (1-n)a_1 = (F_1 - a_1) + na_1 = F_0 + na_1$$

Whence, (114) becomes

$$F_n = F_0 + na_1 + B_1b - C_1c_1 + D_1d - E_1e_1 + \dots \quad (115)$$

which, subtracted from (113), gives

$$0 = (B - B_1)b + (C + C_1)c_1 + (D - D_1)d + \dots \quad (116)$$

The equation (116) is true in all cases to which the formulae of interpolation are applicable; it is therefore true when  $F(T)$  is a rational integral function of the second degree. But, in the latter case, the second differences being constant, we have

$$c_1 = d = e_1 = \dots = 0$$

The equation (116) then becomes

$$0 = (B - B_1)b$$

Hence, since  $b$  cannot vanish, we have

$$B_1 = +B$$

This result reduces (116) to the form

$$0 = (C+C_1)c_1 + (D-D_1)d + (E+E_1)e_1 + \dots \quad (117)$$

Again, we may suppose  $\Delta'''$  constant; that is, we may put

$$d = e_1 = \dots = 0$$

The equation (117) then becomes

$$0 = (C+C_1)c_1$$

or

$$C_1 = -C$$

By repeated application of this reasoning, we prove that the relations (112) are true generally.

It follows that the numerical process involved in finding  $F'_n$  by BESSEL'S Formula is identical whether we interpolate forward from  $F'_0$  or backward from  $F'_1$ , except for the terms in  $F'$  and  $\Delta'$ . Hence little or no check is afforded by performing the interpolation by both methods. When such a check is deemed necessary, BESSEL'S and STIRLING'S Formulae should both be used.

42. *Relative Advantages of NEWTON'S, STIRLING'S, and BESSEL'S Formulae.*—In practice, the only important application of NEWTON'S Formula consists in interpolating functional values near the *beginning* or *end* of a given series. The selection of this formula is then a matter of necessity rather than of preference.

In all other cases, either of the more rapidly converging formulae of STIRLING or BESSEL should be employed. Regarding a choice between these two, when Tables II and III are available there would appear to be very little advantage one way or the other. The form given by BESSEL is more commonly used, and is perhaps a trifle more accurate in practice than STIRLING'S form, particularly for values of  $n$  in the neighborhood of *one-half*. When  $n$  is quite small, however, STIRLING'S Formula will probably be found more convenient.

Suppose we have given a limited table of functions, as follows :

$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$F_{-2}$	$a''$			
$F_{-1}$	$a'$	$b'$		
$F_0$	$a_1$	$b_0$	$c'$	$d_0$
$F_1$	$a_2$	$b_1$	$c_1$	$d_1$
$F_2$	$a_3$	$b_2$	$c_2$	
$F_3$				

Assuming that fourth differences must be taken into account, and that fifth differences are to be neglected, the value of  $F_n$  should in this case be computed by BESSEL'S Formula, which employs the mean of the quantities  $d_0$  and  $d_1$ . If, however, the function  $F_3$  were not included in this series, then the term  $d_1$  would not be given, and we should proceed by STIRLING'S Formula, which involves  $d_0$  directly.

BESSEL'S Formula is particularly simple and convenient when  $n = \frac{1}{2}$ , that is, when it is required to find the function which falls midway between  $F_0$  and  $F_1$ ; this important case will be fully considered in a later section.

43. *Simple Interpolation.*—When frequent interpolation is required, as in tables of logarithms, trigonometric functions, etc., the interval of the argument is usually chosen sufficiently small that the effect of second differences may be neglected. BESSEL'S Formula gives in this case

$$F_n = F_0 + na_1 \quad (118)$$

To interpolate *backwards* from  $F_0$ , that is, to find  $F_{-n}$ , we obtain from (111a), by neglecting second and higher differences,

$$F_{-n} = F_0 - na' \quad (119)$$

Upon these formulae the process of *simple interpolation* is based. The first difference to be used in either case is the value falling between  $F_0$  and the function toward which the interpolation proceeds.

Frequently, where great accuracy is not required, it is sufficient to obtain  $F_n$  by simple interpolation even when the second differences are considerable. In such a case, supposing that the third differences

are insensible, we observe from BESSEL'S Formula that the error of the approximate value of  $F_n$  will be —

$$\delta F_n = \frac{n(n-1)}{2} \Delta'' \quad (120)$$

The maximum value of  $\frac{n(n-1)}{2}$ , which obtains for  $n = \frac{1}{2}$ , is  $-\frac{1}{8}$ ; whence we have the following result:

*When second differences are sensibly constant, the maximum error of functions obtained by simple interpolation is  $\frac{1}{8} \Delta''$ .*

Thus, in Tables I, II, and III, the values of the coefficients for  $\Delta''$  (designated above as  $B$ ) can never be in error by more than  $\frac{1}{8}$  of 10 units, or 1.2 units in the fifth decimal, when found by simple interpolation.

44. *Interpolation Involving Second Differences, by Means of a Corrected First Difference.*—When the second differences are constant, or nearly so, but too large to neglect, their effect may be included (and hence an accurate value of  $F_n$  obtained) by the following simple method:

Since third differences are supposed insensible, BESSEL'S Formula becomes

$$F_n = F_0 + na_1 + \frac{n(n-1)}{2} b$$

which may be written in the form

$$F_n = F_0 + n \left[ a_1 - \left( \frac{1-n}{2} \right) b \right] \quad (121)$$

Now, because third differences are negligible, we may write  $b_0$  for  $b$  in (121); then, putting

$$\left. \begin{aligned} a_1 &= a_1 - \left( \frac{1-n}{2} \right) b_0 \\ F_n &= F_0 + na_1 \end{aligned} \right\} \quad (122)$$

we have

The value of  $F_n$  is thus obtained almost as readily as in simple interpolation. In forming the quantity  $\frac{1-n}{2}$  (which is simply one-half the complement of  $n$  with respect to unity), only an approximate value of  $n$  is ordinarily required. The value of  $a_1$ , the *corrected first*

*difference*, is thus found by an easy mental process amounting almost to mere inspection.

EXAMPLE.—Find  $(8.2)^2$  from the following values of  $T^2$ :

$T$	$T^2$	$\Delta'$	$\Delta''$
4	16		
7	49	+33	+18
10	100	51	+18
13	169	+69	

Here we have

$$t = 7 \quad n = 0.4 \quad F_0 = 49 \quad a_1 = 51 \quad b_0 = 18$$

Hence, by (122), we find

$$\frac{1-n}{2} = \frac{1-0.4}{2} = 0.3$$

$$a_1 = 51 - (0.3 \times 18) = 45.6$$

$$\therefore F_n = 49 + (0.4 \times 45.6) = 67.24$$

This result is exact, because the second differences are rigorously constant.

45. *Backward Interpolation by Means of a Corrected First Difference.*—From (111a), neglecting differences beyond  $\Delta''$ , we obtain

$$F_{-n} = F_0 - na' + \frac{n(n-1)}{2} \cdot \frac{b_0 + b'}{2} = F_0 - na' + \frac{n(n-1)}{2} b_0$$

or

$$F_{-n} = F_0 - n \left( a' + \frac{1-n}{2} b_0 \right) \quad (123)$$

Hence, if we put

$$\left. \begin{aligned} a' &= a' + \left( \frac{1-n}{2} \right) b_0 \\ F_{-n} &= F_0 - na' \end{aligned} \right\} \quad (124)$$

EXAMPLE.—From HILL'S *Tables of Saturn*, the following perturbations are taken; find the value corresponding to the argument  $T = 30682.38$ .

$T$	$F(T)$	$\Delta'$	$\Delta''$
28800	12.5751		
29760	12.1998	-3753	+70
30720	11.8315	3683	68
31680	11.4700	3615	+63
32640	11.1148	-3552	

Taking  $t = 30720$ , we have

$$\begin{array}{ll} F_0 = 11.8315 & n = \frac{720 - 682.38}{960} = 0.03919 \text{ (backward from } F_0) \\ T = 30682.38 & a' = -3683 \\ \omega = 960 & b_0 = + 68 \end{array}$$

Using 0.04 as a sufficiently accurate value of  $n$  in determining  $a'$ , we find by (124),

$$\begin{aligned} \frac{1-n}{2} &= \frac{1-0.04}{2} = 0.48 \\ a' &= -3683 + (0.48 \times 68) = -3650 \\ \therefore F_{-n} &= 11.8315 - [0.03919 \times (-3650)] = 11.8458 \end{aligned}$$

In the present example the algebraic signs of the several quantities of (124) have each been considered. Now it is important to remark that in the majority of cases no attention need be given to these signs; for in this fact lies the chief practical advantage of the method. Thus, in the present example, we are interpolating from the third function toward the second; the value of  $A'$  to be corrected is the difference of these two functions, or 3683; *the sign we disregard*. The correction to be applied to this number is  $0.48 \times 68$ , or 33. Again neglecting signs, we simply apply this quantity to 3683 in such a manner as to obtain a result falling somewhere *between* the numbers 3683 and 3615 of the column  $A'$ . Hence, we *decrease* 3683 by 33, thus obtaining 3650 for our corrected first difference,  $a'$ . Finally,  $na' = 143$ , by which amount we *increase* the function 11.8315 (giving 11.8458), since we observe that the functions are *increasing* in the direction of the interpolation.

A partial exception to this mechanical method of procedure is to be observed when  $a_1$  and  $a'$  have opposite signs; that is, when  $A'$  changes sign in passing the function  $F_0$ . In this case the sign of  $a$  must be noted; we then have, as in (122) and (124),

$$\left. \begin{array}{l} F_n = F_0 + na_1 \\ F_{-n} = F_0 - na' \end{array} \right\} \quad (124a)$$

For example, given the values below :

$T$	$F(T)$	$\Delta'$	$\Delta''$
10	138		
15	538	+400	-300
20	638	+100	-300
25	438	-200	

Suppose it is required to find  $F$ , for  $T=19$ . We let  $t=20$ ,  $F_0=638$ , and interpolate *backwards* with  $n=0.20$ . To obtain  $a'$ , decrease 100 by  $0.4 \times 300$ , or 120; whence  $a'=-20$ , and therefore

$$F_{-n} = F_0 - na' = 638 - [0.2 \times (-20)] = 642$$

We remark in passing that the value of the *corrected first difference*, either in forward or backward interpolation, is always contained between the limits  $a_1$  and  $a'$ .

The number of instances in practice where the differences beyond  $\Delta''$  may be neglected is very large. The precepts given above are therefore important, and should be practiced by the student until their application becomes rapid and mechanical.

46. *Correction of Erroneous Functions by Direct Interpolation of the Values in Question.*—When an error has been detected in some one function of a series by the method of differences, as explained in §8, it is often possible to find the true value of that quantity by direct interpolation. To accomplish this, we have only to omit from the given series every *alternate* function, the incorrect value being one of the number rejected. We have then to make but one interpolation, *midway between* two functions of the new series, to obtain the value required. It is necessary, however, that the given series shall include a sufficient number of functions to furnish an adequate schedule of differences in the abridged table; furthermore, the interval of the original table must be sufficiently small that the magnified differences of the abridged table will not be so large as to render interpolation impossible.

We illustrate by means of Example III, §9. The value of  $\beta$  for May 11.0 was found to be incorrect; hence, to find the true value, we omit from the given series the positions for every noon, retaining

only the values for each midnight. Thus we obtain the following abridged series :

Date 1898	$\beta$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
May 8.5	$-1^{\circ} 59' 54.2''$	$+1^{\circ} 15' 27.2''$			
9.5	$-0^{\circ} 44' 27.0''$	$1^{\circ} 17' 6.9''$	$+1^{\circ} 39.7''$		
10.5	$+0^{\circ} 32' 39.9''$	$1^{\circ} 13' 32.5''$	$-3^{\circ} 34.4''$	$-5^{\circ} 14.1''$	$+54.8$
11.5	$1^{\circ} 46' 12.4''$	$+1^{\circ} 5' 38.8''$	$-7^{\circ} 53.7''$	$-4^{\circ} 19.3''$	
12.5	$+2^{\circ} 51' 51.2''$				

The value of  $\beta$  for May 11.0 is now readily found by interpolation ; for this purpose, we take

$$t = \text{May } 10.5 \qquad F_0 = +0^{\circ} 32' 39''.9 \qquad n = 0.50$$

Since but one value of  $\Delta^{iv}$  is given, namely  $d_0 = 54.8$ , we proceed by STIRLING'S Formula (see §42) ; thus we find

$$\begin{array}{lll}
 A = +\frac{1}{2} & a = +1^{\circ} 15' 19.7'' & F_0 = +0^{\circ} 32' 39''.9 \\
 B = +\frac{1}{8} & b_0 = -3^{\circ} 34.4'' & Aa = +0^{\circ} 37' 39.85'' \\
 C = -\frac{1}{16} & c = -4^{\circ} 46.7'' & Bb_0 = -26.80 \\
 D = -0.00781 & d_0 = +54.8 & Cc = +17.92 \\
 & & Dd_0 = -0.43
 \end{array}$$

$$\therefore \beta (\text{May } 11.0, 1898) = +1^{\circ} 10' 10.44''$$

The value found in §9 by the method of differences is  $+1^{\circ} 10' 10''.6$ . The result just obtained by interpolation is uncertain within narrow limits, because we have no knowledge of the value of  $\Delta^v$  in the above table. The value  $1^{\circ} 10' 10''.6$  should therefore be taken as the more probable.

Had the value of  $\beta$  for May 13.5 been included in the original series, our abridged table would have yielded two values of  $\Delta^{iv}$  and one of  $\Delta^v$ . We should then have used BESSEL'S Formula (see §42) to compute the latitude for May 11.0. Now, the moon's latitude for May 13.5, 1898, is  $+3^{\circ} 46' 22''.2$ ; including this value with the others above, and applying BESSEL'S Formula, we find  $\beta = +1^{\circ} 10' 10''.57$ .

47. When a series contains *several* incorrect functions, separated from each other by *even* multiples of the interval  $\omega$ , the foregoing

method at once serves for the determination of the several values in question. Thus, in the series

$$F_0, F_1, F_2, F_3, F_4, \dots$$

let us suppose that  $F_1$ ,  $F_3$ , and  $F_7$  are in error. Then, if we tabulate and difference the series

$$F_0, F_2, F_4, F_6, F_8, \dots$$

the required values are easily found by interpolation.

Again, when two *adjacent* functions, say  $F_4$  and  $F_5$ , require correction, we may proceed by tabulating every *third* function of the given series; thus we obtain the abridged series

$$F_0, F_3, F_6, F_9, \dots$$

from which the values of  $F_4$  and  $F_5$  are found by interpolating with  $n = \frac{1}{3}$  and  $\frac{2}{3}$ , respectively. Otherwise, if the differences of the latter series are too large for accurate interpolation, we may omit from the original table every *alternate* function only, as in §46. The resulting series,

$$F_0, F_2, F_4, F_6, F_8, \dots$$

will therefore contain but one incorrect value, namely  $F_4$ . The correction to  $F_4$  may then be found by the method of differences, whereas this method might be impracticable if applied to  $F_4$  and  $F_5$  simultaneously. Similarly, we may correct  $F_5$  by the differences of

$$F_1, F_3, F_5, F_7, F_9, \dots$$

or, by interpolation from the corrected series

$$F_0, F_2, F_4, F_6, F_8, \dots$$

#### SYSTEMATIC INTERPOLATION—SUBDIVISION OF TABLES.

48. Thus far we have considered interpolation as a process for computing the values of functions for occasional or *special* values of the argument, simply. We shall now consider the subject in a broader

sense, and find that interpolation is of great importance as applied in a more extended and systematic manner.

When a complicated function is to be computed and tabulated for a large number of equidistant values of the argument, or when the tabular quantities result from a long and laborious calculation, it will be much shorter and easier to make the direct computation for a less frequent interval than is finally required, and thence to obtain the intermediate values by systematic interpolation. For example, suppose the function

$$F(T) = 700''.43 \sin 2T - 1''.19 \sin 4T$$

is to be tabulated for every  $10'$  from  $30^\circ$  to  $60^\circ$ ; we should begin by computing  $F(T)$  for every 4th degree of  $T$ . Thus we should obtain the values of  $F(T)$  for  $T =$

$$22^\circ, 26^\circ, 30^\circ, 34^\circ, . . . . 70^\circ;$$

the calculation being extended somewhat beyond the assigned limits in order to facilitate the interpolation which follows. These quantities having been differenced, and corrected for accidental errors if necessary, the *middle terms* are then found by interpolation to *halves*. We thus obtain the series  $F(T)$  corresponding to  $T =$

$$26^\circ, 28^\circ, 30^\circ, 32^\circ, . . . . 64^\circ$$

Interpolating again to halves, we have a table of  $F(T)$  for every degree of  $T$ . A third interpolation to halves gives the function for every  $30'$ . Finally, interpolating the latter series to *thirds*, we obtain the required table, giving  $F(T)$  for every  $10'$  of the argument  $T$ . It is obvious that the labor of computation decreases rapidly with each successive interpolation.

All of the extended tables in common use, such as tables of logarithms, sines, tangents, etc., have been subdivided in this manner, at a saving of labor almost beyond estimation. In fact, interpolation has undoubtedly done more for mathematical science than any other discovery, excepting that of logarithms.

The following sections will be devoted to the derivation of formulae and precepts which will simplify the process of systematic interpolation

just described. Instead of performing a separate and distinct calculation for each interpolated function, we shall develop a method by which the required values are obtained by *successive additions* of the *computed differences* of those values.

The most convenient interpolation to perform, either in an isolated case, or as applied to the subdivision of an extended series, is interpolation *to halves*, which gives the function corresponding to the *mean* of two consecutive tabular values of the argument. This case will now be considered.

49. *Interpolation to Halves.*—If, in BESSEL'S Formula (111), we put  $n = \frac{1}{2}$ , the coefficients of  $\Delta'''$  and  $\Delta^v$  vanish, and we get

$$F_{\frac{1}{2}} = F_0 + \frac{1}{2}a_1 - \frac{1}{8}b + \frac{3}{128}d - \dots \quad (125)$$

Since  $F_1 - F_0 = a_1$ , we have

$$F_0 + \frac{1}{2}a_1 = \frac{F_0 + F_1}{2}$$

Also, by (106), we have

$$b = \frac{b_0 + b_1}{2}$$

$$d = \frac{d_0 + d_1}{2}$$

Hence, (125) may be written in the form

$$F_{\frac{1}{2}} = \frac{F_0 + F_1}{2} - \frac{1}{8}\left(\frac{b_0 + b_1}{2}\right) + \frac{3}{128}\left(\frac{d_0 + d_1}{2}\right) - \dots \quad (126)$$

which is the formula for *interpolation to halves*, true to fifth differences inclusive. The differences are to be taken according to the schedule on page 62.

Supposing that fourth differences are so small as to produce no sensible effect, we obtain from (126) the very simple formula

$$F_{\frac{1}{2}} = \frac{F_0 + F_1}{2} - \frac{1}{8}\left(\frac{b_0 + b_1}{2}\right) \quad (127)$$

true to third differences inclusive. Hence, to interpolate a function *midway* between two consecutive tabular values, we have the following

RULE : *From the mean of the two given functions, subtract one-eighth the mean of the second differences which stand opposite. The result is true to third differences inclusive. To obtain the value true to fifth differences inclusive, add to the above result  $\frac{3}{128}$  of the mean of the corresponding fourth differences.*

50. *Precepts for Systematic Interpolation to Halves.*—The foregoing rule applies either to the interpolation of a single function into the middle, or to that of an entire series of values. For the latter purpose, however, the work may be arranged in a more expeditious manner, as follows :

For convenience, we assume for the present that 4th differences may be neglected; accordingly, if we put

$\delta_0' = F_{\frac{1}{2}} - F_0 \quad , \quad \delta_1' = F_1 - F_{\frac{1}{2}} \quad , \quad \delta_2' = F_{\frac{3}{2}} - F_1 \quad , \quad \delta_3' = F_2 - F_{\frac{3}{2}} \quad , \quad \dots \quad (128)$   
we obtain from (125),

$\delta_0' = \frac{1}{2}a_1 - \frac{1}{8}\left(\frac{b_0+b_1}{2}\right)$

$\delta_2' = \frac{1}{2}a_2 - \frac{1}{8}\left(\frac{b_1+b_2}{2}\right)$

$\delta_4' = \frac{1}{2}a_3 - \frac{1}{8}\left(\frac{b_2+b_3}{2}\right)$

$\dots \dots \dots$

}

(129)

The quantities  $\delta'$  defined by (128) are evidently the *first differences* of the *interpolated* series; the alternate terms,  $\delta_0', \delta_2', \delta_4', \dots$ , are computed by (129) from the first and second differences of the *given* series of functions; the values of  $\delta_1', \delta_3', \delta_5', \dots$  are not computed. The method and arrangement of the work are shown in the schedule below :

$T$	$F(T)$	$\delta'$	$\delta''$	$\alpha$	$\beta$	$\Delta'$	$\Delta''$	$\Delta'''$
$t - \omega$	$F_{-1}$							
						$a'$		$c'$
$t$	$F_0$						$b_0$	
$t + \frac{1}{2}\omega$	$F_{\frac{1}{2}}$	$\delta_0'$	$\delta_0''$	$\frac{1}{2}a_1$	$-\frac{1}{8}\left(\frac{b_0+b_1}{2}\right)$	$a_1$		$c_1$
$t + \omega$	$F_1$	$\delta_1'$	$\delta_1''$				$b_1$	
$t + \frac{3}{2}\omega$	$F_{\frac{3}{2}}$	$\delta_2'$	$\delta_2''$	$\frac{1}{2}a_2$	$-\frac{1}{8}\left(\frac{b_1+b_2}{2}\right)$	$a_2$		$c_2$
$t + 2\omega$	$F_2$	$\delta_3'$	$\delta_3''$				$b_2$	
$t + \frac{5}{2}\omega$	$F_{\frac{5}{2}}$	$\delta_4'$	$\delta_4''$	$\frac{1}{2}a_3$	$-\frac{1}{8}\left(\frac{b_2+b_3}{2}\right)$	$a_3$		$c_3$
$t + 3\omega$	$F_3$	$\delta_5'$	$\delta_5''$				$b_3$	

The differences of the given series are placed in the last three columns, under  $\Delta'$ ,  $\Delta''$ , and  $\Delta'''$ . The column  $\alpha$  is then filled in by writing opposite each of the quantities  $\Delta'$  one-half its value. The column  $\beta$  is also computed, each term being *minus* one-eighth the mean of the two values of  $\Delta''$  which stand opposite. The *alternate* quantities of column  $\delta'$  are then found, as in (129), by taking the sums of the corresponding terms in  $\alpha$  and  $\beta$ ; the results are written immediately *above* the line of the latter terms, so as to fall between  $F_0$  and  $F_{\frac{1}{2}}$ ,  $F_1$  and  $F_{\frac{3}{2}}$ , etc., respectively.

Finally, since by (128) we have

$$F_{\frac{1}{2}} = F_0 + \delta_0' \quad , \quad F_{\frac{3}{2}} = F_1 + \delta_2' \quad , \quad F_{\frac{5}{2}} = F_2 + \delta_4' \quad , \quad . . . . \quad (130)$$

it is only necessary to add each computed value of  $\delta'$  to the function immediately preceding, to obtain the required middle functions. Having thus completed the interpolation, the remaining or alternate values of  $\delta'$  are filled in by direct differencing. The second differences are then written in the column  $\delta''$ , their regularity proving the accuracy of the work.

The *given functions*, also the *computed* first differences, etc., are distinguished in the above schedule by heavy type.

When it is necessary to take account of 4th and 5th differences, we have only to form an extra column  $\gamma$ , to follow  $\beta$  in the schedule above. Under  $\gamma$  we write the terms

$$\frac{3}{128} \left( \frac{d_0 + d_1}{2} \right) \quad , \quad \frac{3}{128} \left( \frac{d_1 + d_2}{2} \right) \quad , \quad \text{etc.};$$

the values of  $\delta'$  are then formed by adding the three corresponding terms in  $\alpha$ ,  $\beta$ , and  $\gamma$ .

EXAMPLE.—Given the values of  $\log \sin T$  for  $T = 30^\circ, 32^\circ, 34^\circ, \dots, 42^\circ$ ; find the value for every degree of  $T$  from  $32^\circ$  to  $40^\circ$ , inclusive.

In accordance with the method above outlined, we arrange the given functions, with their differences, as follows:

<i>T</i>	Log sin <i>T</i>	$\delta'$	$\delta''$	$\alpha$	$\beta$	$\Delta'$	$\Delta''$	$\Delta'''$
<b>30</b>	<b>9.69897</b>							
31						+2524		
<b>32</b>	<b>9.72421</b>	+1190					-189	
33	9.73611	1145	-45	+1167.5	+22.4	2335	169	+20
<b>34</b>	<b>9.74756</b>	<b>1103</b>	42					
35	9.75859	1063	40	<b>1083.0</b>	<b>20.2</b>	2166		15
<b>36</b>	<b>9.76922</b>	<b>1024</b>	39				154	
37	9.77946	988	36	<b>1006.0</b>	<b>18.3</b>	2012		15
<b>38</b>	<b>9.78934</b>	<b>953</b>	35				139	
39	9.79887	+ 920	-33	+ 936.5	+16.7	1873		+10
<b>40</b>	<b>9.80807</b>						-129	
41						+1744		
<b>42</b>	<b>9.82551</b>							

Since 4th differences may be neglected, only the two columns  $\alpha$  and  $\beta$  are required for the computation of the differences  $\delta'$ . All the quantities actually used in the process are given in the above table. The computed quantities, together with the given values of  $\log \sin T$ , are printed in heavy type, to render this process more evident.

51. *To Reduce the Argument Interval of a Given Table from  $\omega$  to  $m\omega$ , where  $\frac{1}{m}$  is a Positive Odd Integer.*—As particular cases of this problem, we may take  $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{9}$ , etc. Taking  $m = \frac{1}{3}$ , we introduce *two* values between every two adjacent functions of the given table; we thus derive the series

$$F_0, F_{\frac{1}{3}}, F_{\frac{2}{3}}, F_1, F_{\frac{4}{3}}, \dots$$

in which the interval is  $\frac{1}{3}\omega$ . This process is called *interpolation to thirds*. To interpolate *to fifths*, we let  $m = \frac{1}{5}$ , thus introducing *four* functions between every two adjacent terms of the original series. We then have the tabular values of

$$F_0, F_{\frac{1}{5}}, F_{\frac{2}{5}}, F_{\frac{3}{5}}, F_{\frac{4}{5}}, F_1, F_{\frac{6}{5}}, \dots$$

the interval being  $\frac{1}{5}\omega$ .

More generally, let us take  $m = \frac{1}{k}$ , where  $k$  is a positive odd integer; we thus introduce  $k-1$  equidistant values of the function between every two adjacent terms of the given series. The resulting series will therefore be

$$F_0, F_m, F_{2m}, F_{3m}, \dots, F_{(k-1)m}, F_1, F_{1+m}, \dots$$

in which the argument interval is  $m\omega$ , or  $\frac{\omega}{k}$ . Now, the two adjacent functions of this interpolated series, which, as a pair, fall *midway* between  $F_0$  and  $F_1$ , are

$$F_{\left(\frac{k-1}{2}\right)_m} \quad \text{and} \quad F_{\left(\frac{k+1}{2}\right)_m}$$

that is

$$F_{\left(\frac{1-m}{2}\right)} \quad \text{and} \quad F_{\left(\frac{1+m}{2}\right)}$$

Hence, if we put

$$\delta_i' = F_{\left(\frac{1+m}{2}\right)} - F_{\left(\frac{1-m}{2}\right)} \quad (131)$$

it follows that  $\delta_i'$  is the value of the *first difference* of the *interpolated* series which falls on the line *midway* between  $F_0$  and  $F_1$ ; we shall designate this quantity a *middle first difference* of the required series. If we now let

$$\frac{1+m}{2} = n \quad (132)$$

we have

$$\frac{1-m}{2} = 1-n$$

and (131) becomes

$$\delta_i' = F_n - F_{1-n} \quad (133)$$

Hence, to express  $\delta_i'$  in terms of the differences of the *given* series, we have only to express the values of  $F_n$  and  $F_{1-n}$  by BESSEL'S Formula; thus, abbreviating coefficients, we have, as in (113),

$$F_n = F_0 + na_1 + Bb + Cc_1 + Dd + Ee_1 + \dots \quad (134)$$

Also, by virtue of the property of these coefficients established in §41, we have

$$F_{1-n} = F_0 + (1-n)a_1 + Bb - Cc_1 + Dd - Ee_1 + \dots \quad (135)$$

The difference of these equations gives

$$\delta_i' = F_n - F_{1-n} = (2n-1)a_1 + 2Cc_1 + 2Ee_1 + \dots \quad (136)$$

Now, by (132), we have

$$n = \frac{1+m}{2}$$

hence, from (111), we find

$$C = \frac{1}{6} n(n-1)(n-\frac{1}{2}) = \frac{m}{48} (m^2-1)$$

$$E = \frac{1}{120} (n+1)n(n-1)(n-2)(n-\frac{1}{2}) = \frac{1}{240} (n+1)(n-2)C = \frac{m}{3840} (m^2-1)(m^2-9)$$

Substituting these values of  $n$ ,  $C$ , and  $E$  in (136), we obtain the formula

$$\delta_{\frac{1}{2}}' = ma_1 + \frac{m}{24}(m^2-1)c_1 + \frac{m}{1920}(m^2-1)(m^2-9)e_1 + \dots \quad (137)$$

by which the *middle first differences* may be computed in any case, provided  $\frac{1}{m}$  is a positive odd integer.

Let us now consider the schedule below :

$T$	$F(T)$	$\delta'$	$\delta''$	$\delta'''$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$t-\omega$	$F_{-1}$		$\delta_{-1}''$			$b'$		$d'$	
$t-\omega+m\omega$									
		$\delta_{-\frac{1}{2}}'$		$\delta_{-\frac{1}{2}}'''$	$a'$		$c'$		$e'$
$t-m\omega$									
$t$	$F_0$		$\delta_0''$			$b_0$		$d_0$	
$t+m\omega$									
		$\delta_{\frac{1}{2}}'$		$\delta_{\frac{1}{2}}'''$	$a_1$		$c_1$		$e_1$
$t+\omega-m\omega$									
$t+\omega$	$F_1$		$\delta_1''$			$b_1$		$d_1$	

The quantities are here arranged in a manner somewhat similar to the schedule of §50. The given functions,  $F_{-1}$ ,  $F_0$ ,  $F_1$ , . . . . , are separated, successively, by  $k-1$  blank lines or spaces, for the subsequent entry of the interpolated values. The columns  $\delta'$ ,  $\delta''$ , and  $\delta'''$  are also reserved for the differences of the interpolated series; and the differences of the given functions are written to the right, in columns  $\Delta'$  to  $\Delta^v$ .

The value of  $\delta_{\frac{1}{2}}'$  is now computed by (137) from the differences  $a_1$ ,  $c_1$ , and  $e_1$ , which stand opposite. In like manner,  $\delta_{-\frac{1}{2}}'$  is computed from the differences  $a'$ ,  $c'$ , and  $e'$ ;  $\delta_{\frac{1}{2}}'$ , from  $a_2$ ,  $c_2$ , and  $e_2$ ; and so on. We thus obtain a series of *middle first differences*, which are tabulated under  $\delta'$  in the schedule above.

Now it is clear that if we should interpolate the  $k-1$  intermediate terms between  $\delta_{-\frac{1}{2}}'$  and  $\delta_{\frac{1}{2}}'$ , between  $\delta_{\frac{1}{2}}'$  and  $\delta_{\frac{3}{2}}'$ , etc., the resulting series would constitute the consecutive first differences of the *interpolated* series  $F(T)$ ; the required functions would then be formed by successive additions of these differences. The problem of

interpolating the given series  $F(T)$  is thus virtually reduced to that of interpolating the *computed* values of  $\delta'$  in *precisely the same manner*.

Now, let  $\delta_0''$  denote the second difference of the *interpolated* series  $F'$ , which stands opposite  $F_0$ ;  $\delta_1''$ , the second difference opposite  $F_1$ ; etc. It follows that  $\delta_0''$  is the *middle first difference* of the *interpolated* series  $\delta'$ , which falls between  $\delta'_{-1}$  and  $\delta'_1$ ;  $\delta_1''$ , that falling between  $\delta'_1$  and  $\delta'_2$ ; and so on. Hence, we may find  $\delta_0'', \delta_1'', \delta_2'', \dots$  from the computed series  $\delta'_{-1}, \delta'_1, \delta'_2, \dots$ , in precisely the manner that the latter quantities are derived from  $F_{-1}, F_0, F_1, \dots$ ; that is, by application of the general formula (137), *mutatis mutandis*. For this purpose, we must form the differences of the computed series

$$\delta'_{-1}, \delta'_1, \delta'_2, \dots$$

Accordingly, let us put, for brevity,

$$M = \frac{m}{24}(m^2-1) \quad , \quad M' = \frac{m}{1920}(m^2-1)(m^2-9) \tag{138}$$

and (137) becomes

$$\delta_i' = ma_1 + Mc_1 + M'e_1 \tag{139}$$

provided differences beyond  $A^v$  are disregarded. We now form a table of the quantities  $\delta'_{-1}, \delta'_1, \delta'_2, \dots$ , and their differences, as follows :

Function, = $\delta'$	1st Diff.	2d Diff.	3d	4th
$\delta'_{-1} = ma' + Mc' + M'e'$	$mb' + Md'$	$mc' + Me'$	$md'$	$me'$
$\delta_1' = ma_1 + Mc_1 + M'e_1$	$mb_0 + Md_0$	$mc_1 + Me_1$	$md_0$	$me_1$
$\delta_2' = ma_2 + Mc_2 + M'e_2$	$mb_1 + Md_1$	$mc_2 + Me_2$	$md_1$	$me_2$

Whence, applying the general formula (139) to the quantities of this table, we obtain

$$\delta_0'' = m(mb_0 + Md_0) + M(md_0) = m^2b_0 + 2Mmd_0$$

or, by (138),

$$\delta_0'' = m^2b_0 + \frac{m^2}{12}(m^2-1)d_0 \tag{140}$$

by which the quantities  $\delta''_{-1}, \delta_0'', \delta_1'', \dots$  of the former schedule are computed from the differences  $A''$  and  $A^iv$  which stand opposite.

Again, we may suppose that the intermediate values of  $\delta''$  have been interpolated between the computed values  $\delta''_{-1}, \delta_0'', \delta_1'', \dots$ ; this completed series  $\delta''$  constitutes the consecutive second differences

of the *interpolated* series  $F(T)$ . Finally, we shall denote by  $\delta_{\frac{1}{2}}'''$  the third difference of the interpolated series  $F$ , which stands opposite  $\delta_{\frac{1}{2}}'$  in the given schedule. The quantity  $\delta_{\frac{1}{2}}'''$  is therefore the middle first difference of the completed series  $\delta''$ , which falls between  $\delta_0''$  and  $\delta_1''$ ; it bears the same relation to  $\delta_0''$  and  $\delta_1''$ , that  $\delta_{\frac{1}{2}}'$  bears to  $F_0$  and  $F_1$ . Hence, to find  $\delta_{\frac{1}{2}}'''$ , let us put

$$M'' = \frac{m^2}{12}(m^2-1)$$

and (140) becomes

$$\delta_0'' = m^2b_0 + M''d_0 \quad (141)$$

The differences of  $\delta''_{-1}, \delta_0'', \delta_1'', \dots$  are therefore as follows:

Function, = $\delta''$	1st Diff.	2d	3d
$\delta_{-1}'' = m^2b' + M''d'$	$m^2c' + M''e'$	$m^2d'$	$m^2e'$
$\delta_0'' = m^2b_0 + M''d_0$	$m^2c_1 + M''e_1$	$m^2d_0$	$m^2e_1$
$\delta_1'' = m^2b_1 + M''d_1$		$m^2d_1$	

Whence, applying (as above) the general formula (139), we find

$$\delta_{\frac{1}{2}}''' = m(m^2c_1 + M''e_1) + M(m^2e_1) = m^3c_1 + (mM'' + m^2M)e_1$$

Substituting the values of  $M$  and  $M''$ , we have

$$\delta_{\frac{1}{2}}''' = m^3c_1 + \frac{m^3}{8}(m^2-1)e_1 \quad (142)$$

In practice, the values of  $\delta^{\text{iv}}$  and  $\delta^{\text{v}}$  are never required, and in many cases the column  $\delta'''$  is not necessary. Supposing, however, that we have computed the (nearly constant) values of  $\delta_{-\frac{3}{2}}''', \delta_{-\frac{1}{2}}''', \delta_{\frac{1}{2}}''', \dots$  by (142), the intermediate terms are then written in by mere *inspection*. We thus complete the column  $\delta'''$ ,—the consecutive third differences of the required series  $F(T)$ . Having also computed the quantities  $\delta_0'', \delta_1'', \delta_2'', \dots$  and  $\delta'_{-\frac{1}{2}}, \delta'_{\frac{1}{2}}, \delta'_{\frac{3}{2}}, \dots$ , we complete the columns  $\delta''$  and  $\delta'$ , and hence, also, the interpolated series  $F(T)$ , by successive additions.

We now bring together the formulae for  $\delta_{\frac{1}{2}}', \delta_0''$ , and  $\delta_{\frac{1}{2}}'''$ , in the order computed in practice, as follows:

$$\left. \begin{aligned} \delta_{\frac{1}{2}}''' &= m^3c_1 + \frac{m^3}{8}(m^2-1)e_1 \\ \delta_0'' &= m^2b_0 + \frac{m^2}{12}(m^2-1)d_0 \\ \delta_{\frac{1}{2}}' &= ma_1 + \frac{m}{24}(m^2-1)c_1 + \frac{m}{1920}(m^2-1)(m^2-9)e_1 \end{aligned} \right\} \quad (143)$$

which serve to reduce the tabular interval to  $m$  times its original value,  $m$  being the reciprocal of a positive odd integer. It will be observed that the differences required in computing each of the quantities  $\delta$  are always found on the same line with that quantity.

52. *Interpolation to Thirds.*—For this purpose, we take  $m = \frac{1}{3}$  in the formulae (143), and find

$$\left. \begin{aligned} \delta_3''' &= \frac{1}{27} c_1 - \frac{1}{243} e_1 \\ \delta_0'' &= \frac{1}{9} b_0 - \frac{2}{43} d_0, \\ \delta_{\frac{1}{2}}' &= \frac{1}{3} a_1 - \frac{1}{81} c_1 + \frac{1}{729} e_1 \end{aligned} \right\} \tag{144}$$

These formulae are more conveniently computed in the form

$$\left. \begin{aligned} \delta_3''' &= \frac{1}{27} (c_1 - \frac{1}{9} e_1) \\ \delta_0'' &= \frac{1}{9} (b_0 - \frac{2}{7} d_0) \\ \delta_{\frac{1}{2}}' &= \frac{1}{3} (a_1 - \delta_{\frac{1}{2}}''') \end{aligned} \right\} \tag{145}$$

EXAMPLE.—Given the value of  $\log \tan T$  for every third degree of  $T$  from  $27^\circ$  to  $48^\circ$ , inclusive: find the function for every degree between  $33^\circ$  and  $42^\circ$ .

According to the precepts of the last section, we arrange the work as follows:

$T$	$\log \tan T$	$\delta'$	$\delta''$	$\delta'''$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
<b>27</b>	<b>9.70717</b>				+5427			
<b>30</b>	<b>9.76144</b>					−319		
<b>33</b>	<b>9.81252</b>			+3.1	5108		+85	
34	9.82899			3.0		234		−14
35	9.84523	+1646.9	−25.9	2.8				
<b>36</b>	<b>9.86126</b>	<b>1623.8</b>	<b>23.1</b>	<b>2.6</b>	4874		71	
37	9.87711	1603.3	20.5	2.5		163		12
38	9.89281	1585.3	<b>18.0</b>	2.3				
39	9.90837	1569.6	15.7	<b>2.2</b>	4711		59	
40	9.92382	1556.1	13.5	2.0		104		6
41	9.93917	1544.6	<b>11.5</b>	2.0				
<b>42</b>	<b>9.95444</b>	<b>1535.1</b>	9.5	<b>1.9</b>	4607		53	
		+1527.5	7.6	1.9		−51		−2
			−5.7	1.9				
				+1.9	4556		+51	
<b>45</b>	<b>0.00000</b>					0		
					+4556			
<b>48</b>	<b>0.04556</b>							

The heavy type shows at a glance the given functions, and likewise the computed middle differences. We observe that it is here

necessary to compute five values of  $\delta'''$ , four values of  $\delta''$ , and only three of  $\delta'$ . These quantities are computed to one more than the number of decimals given in  $F(T)$ , to avoid accumulation of any appreciable error in the final additions. Having obtained for  $\delta'''$  the series

$$+3.1 \quad 2.6 \quad 2.2 \quad 1.9 \quad +1.9$$

the intermediate terms are readily inserted, as shown above; it is necessary, however, to see that the completed series  $\delta'''$  is consistent with the *computed* values of  $\delta''$ . Thus we must have

$$2.8 + 2.6 + 2.5 = -(18.0 - 25.9) = +7.9$$

$$2.3 + 2.2 + 2.0 = -(11.5 - 18.0) = +6.5$$

$$2.0 + 1.9 + 1.9 = -(5.7 - 11.5) = +5.8$$

If these relations are not satisfied exactly on first trial, the interpolated values of  $\delta'''$  must be adjusted to fulfill the necessary conditions.

The column  $\delta''$  is now completed by successive additions of the quantities  $\delta'''$ . Again, it is necessary to see that the completed series  $\delta''$  agrees with the computed values of  $\delta'$ . For we must have

$$-(20.5 + 18.0 + 15.7) = 1569.6 - 1623.8 = -54.2, \text{ etc.}$$

Since these relations are seldom exact in the beginning, the provisional values of  $\delta''$  will usually require slight alterations.

From the final series  $\delta''$ , we obtain  $\delta'$  by successive additions. As before, an agreement must subsist between the values of  $\delta'$  and the given set of functions; that is, between  $\delta'$  and  $\mathcal{A}'$ . Thus we should have

$$\Sigma \delta' = 1646.9 + 1623.8 + 1603.3 = +4874.0 = \mathcal{A}', \text{ etc.}$$

In the latter case, however, a discrepancy not exceeding four or five units in the added decimal may be tolerated. Our final series  $\delta'$  is therefore satisfactory; whence we obtain by successive additions the required values of  $\log \tan T$ .

53. *Interpolation to Fifths.*—Taking  $m = \frac{1}{5}$  in the formulae (143), we obtain

$$\left. \begin{aligned} \delta_1''' &= \frac{1}{125} (c_1 - \frac{3}{25} e_1) \\ \delta_0'' &= \frac{1}{25} (b_0 - \frac{2}{25} d_0) \\ \delta_1' &= \frac{1}{5} \{ a_1 - \frac{1}{25} (c_1 - \frac{1}{125} e_1) \} \end{aligned} \right\} \quad (146)$$

In practice it will suffice to put  $\frac{1}{9} e_1$  for both  $\frac{3}{25} e_1$  and  $\frac{1}{125} e_1$ ; the formulae (146) then become, very approximately,

$$\left. \begin{aligned} \delta_{\frac{1}{2}}''' &= \frac{1}{125} (c_1 - \frac{1}{9} e_1) \\ \delta_0'' &= \frac{1}{25} (b_0 - \frac{2}{25} d_0) \\ \delta_{\frac{1}{2}}' &= \frac{1}{5} a_1 - \delta_{\frac{1}{2}}''' \end{aligned} \right\}$$

(147)

EXAMPLE.—The following ephemeris gives the moon’s R.A. for every ten hours. Obtain the value for every second hour, from Sept. 23<sup>d</sup> 20<sup>h</sup> to Sept. 25<sup>d</sup> 12<sup>h</sup>, inclusive.

The details of the computation are as follows :

Date, 1898	Moon’s R.A.	$\delta'$	$\delta''$	$\delta'''$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
		m s	s	s	m s	s	s	s
Sept. 23 <sup>d</sup> 0 <sup>h</sup>	18 24 26.4				+25 31.1			
Sept. 23 10	18 49 57.5					−20.1		
				−.034	25 11.0		−4.2	
				32				
				30				
Sept. 23 20	19 15 8.5		−0.976	28		24.3		+1.2
23 22	19 20 7.9	+4 59.39	1.004	26				
24 0	19 25 6.3	58.39	1.030					
24 2	19 30 3.7	4 57.36	1.054	.024	24 46.7		3.0	
24 4	19 35 0.0	56.31	1.077	23				
Sept. 24 6	19 39 55.2	55.23	1.097	20		27.3		1.5
24 8	19 44 49.3	54.13	1.114	17				
24 10	19 49 42.3	53.02	1.128	14				
24 12	19 54 34.2	4 51.89	1.140	.012	24 19.4		1.5	
24 14	19 59 25.0	50.75	1.149	09				
Sept. 24 16	20 4 14.6	49.60	1.156	07		28.8		1.4
24 18	20 9 3.0	48.44	1.161	05				
24 20	20 13 50.3	47.28	1.164	03				
24 22	20 18 36.4	4 46.12	1.165	−.001	23 50.6		−0.1	
25 0	20 23 21.4	44.95	1.163	+.002				
Sept. 25 2	20 28 5.2	43.79	1.160	03		28.9		1.1
25 4	20 32 47.8	42.63	1.154	06				
25 6	20 37 29.3	41.47	1.147	07				
25 8	20 42 9.6	4 40.33	1.139	.008	23 21.7		+1.0	
25 10	20 46 48.8	39.19	1.130	09				
Sept. 25 12	20 51 26.9	+4 38.06	−1.119	11		27.9		+0.9
				12				
				14				
				+.015	22 53.8		+1.9	
Sept. 25 22	21 14 20.7					−26.0		
					+22 27.8			
Sept. 26 8	21 36 48.5							

Here we extend the computation of  $\delta'''$  and  $\delta''$  *two places* of decimals; one of which is dropped in computing  $\delta'$ , and the other in forming the required functions. The principle and method being the same as in the last example, further explanation is unnecessary.

54. *Order of Interpolation to Follow, when a Series Requires Successive Interpolation to Halves, Thirds, etc.*—When a table of functions is to be interpolated, successively, one or more times to *halves*, and also to *thirds* and *fifths*, the easiest method is to proceed in the order named. Thus, if the interval of the original series is  $\omega$ , and that of the final table is  $\omega'$ , we may suppose the relation of these quantities to be—

$$\omega = 2^k \cdot 3^l \cdot 5^m \cdot \omega'$$

where  $k, l$ , and  $m$  are integers. It will then be found most expedient, first, to interpolate to halves,  $k$  times; then to thirds,  $l$  times; and finally to fifths,  $m$  times.

For example,  $F$  being given for every *degree*, and required for every *minute* of arc, we should first interpolate to  $30'$ , then to  $15'$ , then to  $5'$ , and finally to every minute of arc.

55. *To Interpolate with a Constant Interval  $n$ , an Entire Series of Functions.*—Let the given series, with its differences, be as follows:

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$t$	$F_0$		$b_0$		$d_0$
$t + \omega$	$F_1$	$a_1$	$b_1$	$c_1$	$d_1$
$t + 2\omega$	$F_2$	$a_2$	$b_2$	$c_2$	$d_2$
$t + 3\omega$	$F_3$	$a_3$	$b_3$	$c_3$	$d_3$
$t + 4\omega$	$F_4$	$a_4$	$b_4$	$c_4$	$d_4$

It is required to interpolate the values of  $F_n, F_{1+n}, F_{2+n}, F_{3+n}, \dots$ . These functions evidently form a new series having the same interval as the old. Let us denote this new series by  $[F]$ ; also, let the differences of  $[F]$ , denoted by  $[\Delta']$ ,  $[\Delta'']$ ,  $[\Delta''']$ ,  $\dots$ , be taken as shown in the table below:

$T$	$[F]$	$[\Delta']$	$[\Delta'']$	$[\Delta''']$	$[\Delta^{iv}]$
$t + n\omega$	$F_n$		$\beta_0$		$\delta_0$
$t + (1+n)\omega$	$F_{1+n}$	$\alpha_1$	$\beta_1$	$\gamma_1$	$\delta_1$
$t + (2+n)\omega$	$F_{2+n}$	$\alpha_2$	$\beta_2$	$\gamma_2$	$\delta_2$
$t + (3+n)\omega$	$F_{3+n}$	$\alpha_3$	$\beta_3$	$\gamma_3$	$\delta_3$
$t + (4+n)\omega$	$F_{4+n}$	$\alpha_4$	$\beta_4$	$\gamma_4$	$\delta_4$

Now, it was shown in §22 that differences of *any* order may be expressed in terms of the tabular functions. Thus, in particular, we obtain from the given series  $F$ ,

$$\left. \begin{aligned} c_1 &= F_2 - 3F_1 + 3F_0 - F_{-1} \equiv \Psi(t) \\ c_2 &= F_3 - 3F_2 + 3F_1 - F_0 = \Psi(t+\omega) \\ c_3 &= F_4 - 3F_3 + 3F_2 - F_1 = \Psi(t+2\omega) \\ &\dots\dots\dots \end{aligned} \right\} \tag{148}$$

where  $\Psi(t)$  denotes, for brevity, the function of  $t$  expressed by  $F_2 - 3F_1 + 3F_0 - F_{-1}$ ; that is,  $\Psi(t) \equiv F(t+2\omega) - 3F(t+\omega) + 3F(t) - F(t-\omega)$

Again, in like manner, the *interpolated* series  $[F]$  gives

$$\left. \begin{aligned} \gamma_1 &= F_{2+n} - 3F_{1+n} + 3F_n - F_{-1+n} = \Psi(t+n\omega) \\ \gamma_2 &= F_{3+n} - 3F_{2+n} + 3F_{1+n} - F_n = \Psi(t+\omega+n\omega) \\ &\dots\dots\dots \end{aligned} \right\} \tag{149}$$

It follows, then, that the series  $[\Delta''']$  is simply the series  $\Delta'''$  interpolated forward with the constant interval  $n$ . Moreover, since the above reasoning is perfectly general, this relation holds for *any order* of differences.

Hence, to perform the required interpolation of the series  $F(T)$ , that is, to obtain the series  $[F]$ , we have only to interpolate forward each value of  $\Delta''$  with the constant interval  $n$ , thus forming the column  $[\Delta'']$ . This process is obviously brief and simple. Then, if we compute *occasional* values of  $[\Delta']$ , and also of  $[F]$ , we readily complete the required table by successive additions, as in the preceding problems.

EXAMPLE.—To illustrate the process, we tabulate the “Latitude Reduction” for every fourth degree of latitude ( $\varphi$ ) from  $30^\circ$  to  $82^\circ$ , and thence derive the series for  $\varphi = 35^\circ, 39^\circ, 43^\circ, \dots\dots 75^\circ$ . The work is arranged as follows :

$\varphi$	$\varphi - \varphi'$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\varphi$	$[\varphi - \varphi']$	$[\Delta']$	$[\Delta'']$
$30^\circ$	"					$35^\circ$	"		
30	605.56	+43.04				35	<b>657.422</b>		
34	648.60	30.46	-12.58	-0.60	+.24	39	684.634	+27.212	-13.294
38	679.06	17.28	13.18	0.36	.27	43	698.552	13.918	<b>13.587</b>
42	696.34	+ 3.74	13.54	-0.09	.26	47	698.883	+ 0.331	<b>13.612</b>
46	700.08	- 9.89	13.63	+0.17	.27	51	685.602	-13.281	<b>13.376</b>
50	690.19	23.35	13.46	0.44	.26	55	<b>658.945</b>	26.657	<b>12.868</b>
54	666.84	36.37	13.02	0.70	.23	59	619.420	<b>39.525</b>	<b>12.110</b>
58	630.47	48.69	12.32	0.93	.24	63	567.785	51.635	<b>11.118</b>
62	581.78	60.08	11.39	1.17	.19	67	505.032	62.753	<b>9.897</b>
66	521.70	70.30	10.22	1.36	.19	71	432.382	72.650	- 8.488
70	451.40	79.16	8.86	1.55	+.15	75	<b>351.244</b>	-81.138	
74	372.24	86.47	7.31	+1.70					
78	285.77	- 5.61							
82	193.69	-92.08							

Taking  $n = 0.25$ , we compute by BESSEL'S Formula the values of  $\varphi - \varphi'$  for  $\varphi = 35^\circ$ ,  $55^\circ$ , and  $75^\circ$ , extending the decimal one unit. Similarly, we compute *three* values of  $[\Delta']$ , and *all* of  $[\Delta'']$ ; the computed quantities being clearly shown by heavier type. Adjusting slightly the series  $[\Delta'']$  to conform to the *computed* values of  $[\Delta']$ , we complete the latter column by successive additions. The values of  $[\Delta']$  being found to accord with the computed functions, we complete the entire series as required.

Since the computed intermediate values of  $[\Delta']$  and  $[F']$  serve only as checks, it is obvious that their positions, as also the intervals of their distribution, are entirely arbitrary. These are details to be decided by the computer's judgement in any given case.

It may occasionally be practicable to extend the process to the computation of  $[\Delta''']$ .

## EXAMPLES.

1. Tabulate the five-place log cosines of  $15^\circ, 18^\circ, 21^\circ, 24^\circ, 27^\circ, 30^\circ$ ; from these values interpolate  $\log \cos T$  for  $T = 17^\circ 43', 23^\circ 8',$  and  $28^\circ 15',$  respectively.

2. Given the following table :

$T$	$F(T)$	$T$	$F(T)$
10	17.31	40	14.16
20	14.68	50	16.34
30	13.62	60	20.18

Compute the values of  $F$  for  $T = 24.6, 28.8, 32.3,$  and  $48.5,$  using either BESSEL'S or STIRLING'S Formula.

3. Interpolate the required functions of Example 2 by means of a *corrected first difference*, as explained in §§ 44 and 45.

4. What is the maximum error of interpolation in the table of Example 2, supposing that second differences are neglected?

5. Find the correct values of the erroneous functions in the several tables of Example 6, Chap. I, by direct interpolation, as explained in §§ 46 and 47.

6. Given the following twelve-hour ephemeris of lunar distances of *Spica* :

Date 1898	L.D. of <i>Spica</i>	Date 1898	L.D. of <i>Spica</i>
July 1.0 <sup>d</sup>	43 24 9 <sup>o / "</sup>	July 3.0 <sup>d</sup>	73 35 46 <sup>o / "</sup>
1.5	50 52 0	3.5	81 12 52
2.0	58 24 0	4.0	88 48 56
July 2.5	65 59 1	July 4.5	96 22 40

Interpolate the series *twice* to halves; the first result to include the values from July 1<sup>d</sup>.5 to 4<sup>d</sup>.0, and the final three-hour ephemeris to extend from July 2<sup>d</sup> 0<sup>h</sup> to July 3<sup>d</sup> 12<sup>h</sup>, inclusive.

7. The ephemeris below gives the sun's true longitude for every third day:

1898	Sun's Longitude	1898	Sun's Longitude
	<sup>°</sup> <sup>'</sup> <sup>"</sup>		<sup>°</sup> <sup>'</sup> <sup>"</sup>
Oct. 7	194 14 35.2	Oct. 16	203 9 32.9
10	197 12 34.2	19	206 8 29.4
Oct. 13	200 10 54.0	Oct. 22	209 7 42.0

Derive from these values a *daily* ephemeris extending from Oct. 10 to Oct. 19, inclusive.

8. The following table contains the heliocentric longitude of *Jupiter* for every 80th day of 1898–99, beginning with Jan. 0, 1898:

Date 1898	Helioc. Long. of <i>Jupiter</i>	Date 1898	Helioc. Long. of <i>Jupiter</i>
	<sup>°</sup> <sup>'</sup> <sup>"</sup>		<sup>°</sup> <sup>'</sup> <sup>"</sup>
0 <sup>d</sup>	178 59 17.9	320 <sup>d</sup>	203 10 20.5
80	185 2 24.1	400	209 13 53.8
160	191 5 0.3	480	215 18 35.1
240	197 7 30.9	560	221 24 48.1

Interpolate this table to halves, extending the series from 120<sup>d</sup> to 440<sup>d</sup> inclusive; designate this forty-day ephemeris, Table *A*. Then interpolate *A* to fifths, denoting the eight-day series by *B*. Let the limits of *B* be 200<sup>d</sup> and 320<sup>d</sup>, respectively. Retain copies of *A* and *B*.

9. Interpolate (forward) the longitudes of Table *A*, Example 8, with the constant interval  $n = 0.20$ , by the method of §55. This will furnish an ephemeris for the dates 168<sup>d</sup>, 208<sup>d</sup>, 248<sup>d</sup>, . . . . 368<sup>d</sup>. Compare the longitudes thus found for 208<sup>d</sup>, 248<sup>d</sup>, and 288<sup>d</sup>, with their values in Table *B*, Example 8.

10. Deduce from the general formulae (143), the special formulae for interpolation to *sevenths*. Make an application to the five-figure

logarithms of 47, 54, 61, . . . . 96, by computing the logarithms of the consecutive numbers between 61 and 75.

11. Show that if the formulae (143) were extended to include the middle differences of order  $i$ , we should have (using the symbolic form of notation employed in the analogous formulae (64))

$$\begin{aligned}\delta^i &= (\delta)^i = \left( m\Delta + \frac{m}{24}(m^2-1)\Delta^3 + \frac{m}{1920}(m^2-1)(m^2-9)\Delta^5 + \dots \right)^i \\ &= m^i\Delta^i + \frac{im^i}{24}(m^2-1)\Delta^{i+2} + \frac{im^i}{5760}(m^2-1)\{5i-2\}m^2 - (5i+22)\Delta^{i+4} + \dots\end{aligned}$$

in which  $i$  may be either odd or even ; and where  $\Delta^i, \Delta^{i+2}, \Delta^{i+4}, \dots$  symbolize the tabular differences which fall upon the same horizontal line with  $\delta^i$ .

## CHAPTER III.

### DERIVATIVES OF TABULAR FUNCTIONS.

56. It is often required to find certain numerical values of the differential coefficients of functions either analytically unknown, or complicated in expression. In the majority of such cases the function has been previously tabulated for particular (equidistant) values of the argument. The required derivatives are then readily computed from the *differences* of the tabular functions.

We have already seen that—with certain limitations—particular values of a function, with their differences, practically determine the character and law of that function, thus enabling us to determine intermediate values by interpolation. The trend or law of variation of the function being thus defined by its differences, it is but natural to suppose that the successive *derivatives* are quantities closely related to these *differences*; since the derivatives are themselves direct indices of the character of variation of the function.

57. *Practical Applications.*—The most useful application is in finding the change or *variation* in  $F(T)$  corresponding to an increase of *one unit* in  $T$ , supposing the *rate* of change in  $F$  to remain constant from  $T$  to  $T+1$ , and equal to the actual rate at the instant  $T$ ; for this quantity is simply the first differential coefficient of  $F(T)$  with respect to  $T$ , which we shall denote by  $F'(T)$ .

For example, having observed that a freely falling body describes sixteen feet during the first second of its descent, forty-eight feet the second second, and eighty feet the third, its *velocity* at the end of two seconds is easily found to be sixty-four feet per second. This velocity of sixty-four feet is nothing more than the first differential coefficient of the *space* with respect to the *time*, computed for the instant 2<sup>s</sup>.0: it is the space which would be described during the third

second, supposing the action of gravity to have ceased at the end of the second second.

The most frequent and important applications occur in Astronomy. An astronomical ephemeris contains a great variety of tables giving the positions and motions of various heavenly bodies, and of certain points of reference. From the given positions, tabulated for every hour or from day to day, are derived the motions per minute, per hour, or per day, according to circumstances. For instance, the *Nautical Almanac* gives the sun's declination for every Greenwich noon. The *hourly motion* in declination (also given for every noon) is computed from the *differences* of the tabular declinations: its value is the differential coefficient of the tabular function at the date in question.

In the following sections the various formulæ employed in computing the derivatives of tabular functions will be derived.

58. *Development of the Required Formulæ in General Terms.*—The variables  $T$  and  $n$  are connected by the fundamental relation

$$T = t + n\omega \quad (150)$$

in which  $t$  and  $\omega$  are constants for a given series. Accordingly, we have hitherto written, under varying circumstances,

$$F(T) \quad , \quad F(t+n\omega) \quad , \quad F_n$$

as equivalent expressions of the same quantity. In like manner, we shall hereafter denote the successive *derivatives* of  $F(T)$  by the following equivalent forms:

$$\left. \begin{aligned} \frac{d}{dT} \left\{ F(T) \right\} &\equiv F'(T) \equiv F'(t+n\omega) \equiv F'_n \\ \frac{d^2}{dT^2} \left\{ F(T) \right\} &\equiv F''(T) \equiv F''(t+n\omega) \equiv F''_n \\ \frac{d^3}{dT^3} \left\{ F(T) \right\} &\equiv F'''(T) \equiv F'''(t+n\omega) \equiv F'''_n \\ \frac{d^4}{dT^4} \left\{ F(T) \right\} &\equiv F^{iv}(T) \equiv F^{iv}(t+n\omega) \equiv F^{iv}_n \\ &\dots \dots \dots \end{aligned} \right\} \quad (151)$$

When it is convenient to proceed *backwards* from the argument  $t$  with the interval  $n$ , we shall use the expressions

$$F'_{-n} \equiv F'(t-n\omega) \quad , \quad F''_{-n} \equiv F''(t-n\omega) \quad , \quad F'''_{-n} \equiv F'''(t-n\omega) \quad , \quad \dots \dots \dots \quad (152)$$

Now, by means of any one of the fundamental formulae of interpolation, we may express  $F'_n$  in the form

$$F_n = F_0 + na + Bb + Cc + Dd + Ee + \dots \quad (153)$$

where, in any given case,  $a, b, c, \dots$  are known differences; and where  $B, C, D, \dots$  are definite functions of  $n$ . Let the successive derivatives of  $B, C, D, \dots$ , taken with respect to  $n$ , be denoted by

$$\begin{array}{l} B', B'', B''', \dots \\ C', C'', C''', \dots \\ D', D'', D''', \dots \\ E', E'', E''', \dots \\ \dots \end{array}$$

Then, observing that the coefficient of  $\Delta^{(i)}$  is always of the degree  $i$  in  $n$ , we have

$$\left. \begin{array}{llll} \frac{dB}{dn} = B' & \frac{dC}{dn} = C' & \frac{dD}{dn} = D' & \frac{dE}{dn} = E' \\ \frac{d^2B}{dn^2} = B'' & \frac{d^2C}{dn^2} = C'' & \frac{d^2D}{dn^2} = D'' & \frac{d^2E}{dn^2} = E'' \\ \frac{d^3B}{dn^3} = 0 & \frac{d^3C}{dn^3} = C''' & \frac{d^3D}{dn^3} = D''' & \frac{d^3E}{dn^3} = E''' \\ & \frac{d^4C}{dn^4} = 0 & \frac{d^4D}{dn^4} = D^{iv} & \frac{d^4E}{dn^4} = E^{iv} \\ & & \frac{d^5D}{dn^5} = 0 & \frac{d^5E}{dn^5} = E^v \\ & & & \frac{d^6E}{dn^6} = 0 \end{array} \right\} \quad (154)$$

Reverting to (151), we have

$$F'_n = \frac{dF_n}{dT} = \frac{dF_n}{dn} \cdot \frac{dn}{dT} \quad (155)$$

From (150) we derive

$$\frac{dn}{dT} = \frac{1}{\omega} \quad (156)$$

whence

$$F'_n = \frac{1}{\omega} \cdot \frac{dF_n}{dn} \quad (157)$$

In like manner we obtain

$$\left. \begin{aligned} F''_n &= \frac{dF'_n}{dT} = \frac{dF'_n}{dn} \cdot \frac{dn}{dT} = \frac{1}{\omega^2} \cdot \frac{d^2F_n}{dn^2} \\ F'''_n &= \frac{dF''_n}{dT} = \frac{dF''_n}{dn} \cdot \frac{dn}{dT} = \frac{1}{\omega^3} \cdot \frac{d^3F_n}{dn^3} \\ F^{\text{iv}}_n &= \frac{dF'''_n}{dT} = \frac{dF'''_n}{dn} \cdot \frac{dn}{dT} = \frac{1}{\omega^4} \cdot \frac{d^4F_n}{dn^4} \\ F^\text{v}_n &= \frac{dF^{\text{iv}}_n}{dT} = \frac{dF^{\text{iv}}_n}{dn} \cdot \frac{dn}{dT} = \frac{1}{\omega^5} \cdot \frac{d^5F_n}{dn^5} \\ &\dots \end{aligned} \right\} \quad (158)$$

Therefore, using (153) and (154), we find

$$\left. \begin{aligned} F'_n &= \frac{1}{\omega} (a + B'b + C'c + D'd + E'e + \dots) \\ F''_n &= \frac{1}{\omega^2} (B''b + C''c + D''d + E''e + \dots) \\ F'''_n &= \frac{1}{\omega^3} (C'''c + D'''d + E'''e + \dots) \\ F^{\text{iv}}_n &= \frac{1}{\omega^4} (D^{\text{iv}}d + E^{\text{iv}}e + \dots) \\ F^\text{v}_n &= \frac{1}{\omega^5} (E^\text{v}e + \dots) \\ &\dots \end{aligned} \right\} \quad (159)$$

which are the general formulae for computing the derivatives of  $F(T)$  in terms of the tabular differences.

To derive the formulae for  $F'_{-n}$ ,  $F''_{-n}$ ,  $F'''_{-n}$ ,  $\dots$ , that is, to find the successive derivatives of  $F(t-n\omega)$ , we have only to alter slightly certain details of the preceding development, as follows:

(1) For equation (153) must be substituted the corresponding expression for  $F_{-n}$ , which has the form\*

$$F_{-n} = F_0 - n\alpha + B\beta - C\gamma + D\delta - E\epsilon + \dots \quad (160)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\dots$  are, in general, different from the differences  $a$ ,  $b$ ,  $c$ ,  $\dots$  of (153).

(2) In the present case, we have

$$T = t - n\omega$$

and therefore

$$\frac{dn}{dT} = -\frac{1}{\omega}$$

which must be substituted for equation (156) above.

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\* Compare (75), (105) and (111a) with (73), (104) and (111), respectively.

Introducing these changes, and operating as before, we obtain the required formulae, namely,

$$\left. \begin{aligned} F'_{-n} &= \frac{1}{\omega} (a - B'\beta + C'\gamma - D'\delta + E'\epsilon - \dots) \\ F''_{-n} &= \frac{1}{\omega^2} (B''\beta - C''\gamma + D''\delta - E''\epsilon + \dots) \\ F'''_{-n} &= \frac{1}{\omega^3} (C'''\gamma - D'''\delta + E'''\epsilon - \dots) \\ F^{\text{iv}}_{-n} &= \frac{1}{\omega^4} (D^{\text{iv}}\delta - E^{\text{iv}}\epsilon + \dots) \\ F^{\text{v}}_{-n} &= \frac{1}{\omega^5} (E^{\text{v}}\epsilon - \dots) \\ &\dots \end{aligned} \right\} \quad (161)$$

It now remains to apply (159) and (161) specifically to each of the several formulae of interpolation, of which (153) is the general type. It is obvious that a particular set of coefficients,  $B', B'', \dots, C', C'', \dots$ , etc., will result in each case.

59. *To Compute Derivatives of  $F(T)$  at or near the Beginning of a Series.*—The formulae adapted to this purpose are derived from NEWTON'S Formula of interpolation (73), which is —

$$F_n = F_0 + na_0 + Bb_0 + Cc_0 + Dd_0 + Ee_0 + \dots \quad (162)$$

where

$$\left. \begin{aligned} B &= \frac{n(n-1)}{1^2} = \frac{n^2}{2} - \frac{n}{2} \\ C &= \frac{n(n-1)(n-2)}{1^3} = \frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3} \\ D &= \frac{n(n-1)(n-2)(n-3)}{1^4} = \frac{n^4}{24} - \frac{n^3}{4} + \frac{11}{24}n^2 - \frac{n}{4} \\ E &= \frac{n(n-1)\dots(n-4)}{1^5} = \frac{n^5}{120} - \frac{n^4}{12} + \frac{7}{24}n^3 - \frac{5}{12}n^2 + \frac{n}{5} \\ &\dots \end{aligned} \right\} \quad (163)$$

Differentiating these expressions successively with respect to  $n$ , as indicated in (154), and substituting the resulting values of  $B', B'', \dots, C', C'', \dots$ , etc., in the general formulae (159), we obtain

$$\begin{aligned}
 F'(t+n\omega) &= \frac{1}{\omega} \left( a_0 + (n-\frac{1}{2})b_0 + (\frac{n^2}{2} - n + \frac{1}{3})c_0 + (\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4})d_0 \right. \\
 &\quad \left. + (\frac{n^4}{24} - \frac{n^3}{3} + \frac{7}{8}n^2 - \frac{5}{6}n + \frac{1}{5})e_0 + \dots \right) \\
 F''(t+n\omega) &= \frac{1}{\omega^2} \left( b_0 + (n-1)c_0 + (\frac{n^2}{2} - \frac{3}{2}n + \frac{1}{2})d_0 + (\frac{n^3}{6} - n^2 + \frac{7}{4}n - \frac{5}{6})e_0 + \dots \right) \\
 F'''(t+n\omega) &= \frac{1}{\omega^3} \left( c_0 + (n-\frac{3}{2})d_0 + (\frac{n^2}{2} - 2n + \frac{7}{4})e_0 + \dots \right) \\
 F^{iv}(t+n\omega) &= \frac{1}{\omega^4} \left( d_0 + (n-2)e_0 + \dots \right) \\
 F^v(t+n\omega) &= \frac{1}{\omega^5} \left( e_0 + \dots \right) \\
 &\dots
 \end{aligned} \tag{164}$$

These formulae determine the derivatives of  $F(T)$  for any or all values of  $T$  between  $t$  and  $t+\omega$ , according as we assign different values to  $n$ . As in preceding applications,  $n$  is always a positive proper fraction.

When, as is frequently the case, derivatives are required for some *tabular* value of the argument, say  $t$ , we have only to make  $n=0$  in (164); we thus derive the following simple expressions:

$$\begin{aligned}
 F'(t) &= \frac{1}{\omega} (a_0 - \frac{1}{2}b_0 + \frac{1}{3}c_0 - \frac{1}{4}d_0 + \frac{1}{5}e_0 - \dots) \\
 F''(t) &= \frac{1}{\omega^2} (b_0 - c_0 + \frac{1}{2}d_0 - \frac{5}{6}e_0 + \dots) \\
 F'''(t) &= \frac{1}{\omega^3} (c_0 - \frac{3}{2}d_0 + \frac{7}{4}e_0 - \dots) \\
 F^{iv}(t) &= \frac{1}{\omega^4} (d_0 - 2e_0 + \dots) \\
 F^v(t) &= \frac{1}{\omega^5} (e_0 - \dots) \\
 &\dots
 \end{aligned} \tag{165}$$

The differences employed in (164) and (165) must be taken according to the schedule on page 3, as in direct applications of NEWTON'S Formula.

The formulae (165) have already been established in §18; for it will be observed that (45) and (165) are identical, since in the former  $D, D^2, D^3, \dots$  are used symbolically to denote  $\omega F'(t), \omega^2 F''(t), \omega^3 F'''(t), \dots$

Owing to the special practical importance of the *first* derivative, the coefficients of  $F''(t+n\omega)$ , namely,

$$\left. \begin{aligned} B' &= n - \frac{1}{2} & D' &= \frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4} \\ C' &= \frac{n^2}{2} - n + \frac{1}{3} & E' &= \frac{n^4}{24} - \frac{n^3}{3} + \frac{7}{8}n^2 - \frac{5}{6}n + \frac{1}{5} \end{aligned} \right\} \quad (166)$$

have been tabulated in Table IV for every hundredth of a unit in the argument  $n$ . By means of these quantities, we readily compute  $F''(t+n\omega)$  from the formula

$$F'(t+n\omega) = \frac{1}{\omega} (a_0 + B'b_0 + C'e_0 + D'd_0 + E'e_0) \quad (167)$$

The formulae (164), (165), and (167) are especially adapted to the computation of derivatives at or near the beginning of a tabular series. We shall now solve a few examples to illustrate their use.

EXAMPLE I.—From the following table of  $F(T) \equiv 0.3T^4 - 2T^2 + 4$ , compute  $F''(T)$  for  $T = 2.8$ .

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
0	4.0	— 3.2			
2	0.8	+ 48.0	+ 51.2	+172.8	
4	48.8	272.0	224.0	288.0	+115.2
6	320.8	784.0	512.0	+403.2	+115.2
8	1104.8	+1699.2	+915.2		
10	2804.0				

Here we have

$$\begin{array}{llll} t = 2 & \omega = 2 & a_0 = + 48.0 & c_0 = +288.0 \\ T = 2.8 & n = 0.40 & b_0 = +224.0 & d_0 = +115.2 \end{array}$$

Hence, using the second equation of (164), we find

$$\begin{array}{ll} C'' = n - 1 = -0.60 & b_0 = +224.0 \\ D'' = \frac{n^2}{2} - \frac{3}{2}n + \frac{1}{2} = +0.39\frac{2}{3} & C''c_0 = -172.80 \\ & D''d_0 = + 45.696 \\ \hline \therefore \omega^2 F''_n = + 96.896 \end{array}$$

Whence we obtain

$$F''_n = 96.896 \div 4 = +24.224$$

This result is easily verified from the known analytical form of the function; thus, since

$$F(T) = 0.3T^4 - 2T^2 + 4$$

we derive

$$F'(T) = 1.2T^3 - 4T, \quad F''(T) = 3.6T^2 - 4$$

Substituting  $T = 2.8$  in the last equation, we obtain

$$F''(T) = +24.224$$

as found above.

EXAMPLE II. — From the table of the last example, compute  $F'(T)$  for  $T = 0$ .

Here we employ the first of (165). Making  $t = 0$ , we have

$$a_0 = -3.2 \quad b_0 = +51.2 \quad c_0 = +172.8 \quad d_0 = +115.2$$

We therefore obtain

$$F'(t) = \frac{1}{2}(-3.2 - \frac{51.2}{2} + \frac{172.8}{3} - \frac{115.2}{4}) = 0$$

The result is obviously correct; for we have

$$F'(T) = 1.2T^3 - 4T$$

which vanishes for  $T = 0$ .

EXAMPLE III. — Given the following table of  $F(T) \equiv \sin^2 T$ : compute  $F'(T)$  for  $T = 8^\circ 36'$ .

$T$	$F(T) \equiv \sin^2 T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$^\circ$						
4	0.004866					
8	0.019369	+14503	+9355			
12	0.043227	23858	8891	-464		
16	0.075976	32749	8253	638	-174	
20	0.116978	41002	7455	798	160	+14
24	0.165435	48457	+6512	-943	-145	+15
28	0.220404	+54969				

Here we have

$$t = 8^\circ$$

$$T = 8^\circ 36'$$

$$\omega = 4^\circ = \frac{\pi}{45} = 0.069813 +$$

$$n = \frac{36}{4 \times 60} = 0.15$$

Taking the coefficients  $B'$ ,  $C'$ ,  $D'$  and  $E'$  from Table IV with  $n = 0.15$ , and the differences  $a_0$ ,  $b_0$ ,  $c_0$ , . . . . from the given table, we find, in accordance with (167),

		$a_0 = +0.023858$	
$B' = -0.35$	$b_0 = +8891$	$B'b_0 = -$	3111.9
$C' = +0.19458$	$c_0 = -638$	$C'c_0 = -$	124.1
$D' = -0.12881$	$d_0 = -160$	$D'd_0 = +$	20.6
$E' = +0.09358$	$e_0 = +15$	$E'e_0 = +$	1.4
<hr/>		<hr/>	
$\log (\omega F'_n) = 8.314794$		$\therefore \omega F'_n = +0.020644$	
$\log \omega = 8.843937$			
<hr/>		<hr/>	
$\log F'_n = 9.470857$		$\therefore F'_n = +0.295704$	

This result is easily verified by observing that

$$F'(T) = \frac{d}{dT}(\sin^2 T) = \sin 2T$$

which, for  $T = 8^\circ 36'$ , becomes

$$F'(T) = \sin 17^\circ 12' = 0.295708$$

The former value is thus seen to be very nearly exact.

If the variation in  $F(T)$  corresponding to an increase of *one degree* in  $T$  were required in the present example, the result would be, simply,

$$F'(T) = 0.020644 \div 4 = +0.005161$$

60. *To Compute Derivatives of  $F(T)$  at or near the End of a Series.*—In this case the requisite formulae are derived from NEWTON'S Formula for *backward* interpolation (75), namely,

$$F_{-n} = F_0 - na_{-1} + Bb_{-2} - Cc_{-3} + Dd_{-4} - Ee_{-5} + \dots \quad (168)$$

where  $B, C, D, \dots$  have the values given by (163), as before; and where the differences  $a_{-1}, b_{-2}, c_{-3}, \dots$  are taken according to the schedule below :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$t - 5\omega$	$F_{-5}$		$b_{-6}$		$d_{-7}$	
$t - 4\omega$	$F_{-4}$	$a_{-5}$	$b_{-5}$	$c_{-6}$	$d_{-6}$	$e_{-7}$
$t - 3\omega$	$F_{-3}$	$a_{-4}$	$b_{-4}$	$c_{-5}$	$d_{-5}$	$e_{-6}$
$t - 2\omega$	$F_{-2}$	$a_{-3}$	$b_{-3}$	$c_{-4}$	$d_{-4}$	$e_{-5}$
$t - \omega$	$F_{-1}$	$a_{-2}$	$b_{-2}$	$c_{-3}$		
$t$	$F_0$	$a_{-1}$				

Comparing (168) with the general formula (160), we have

$$\alpha = a_{-1} \quad , \quad \beta = b_{-2} \quad , \quad \gamma = c_{-3} \quad , \quad . . . .$$

Therefore, substituting the previously determined values of  $B'$ ,  $B''$ , . . . .,  $C'$ ,  $C''$ , . . . ., etc., in the general formulae (161), we obtain

$$\left. \begin{aligned} F' (t-n\omega) &= \frac{1}{\omega} \left( a_{-1} - (n-\frac{1}{2}) b_{-2} + (\frac{n^2}{2} - n + \frac{1}{3}) c_{-3} - (\frac{n^3}{6} - \frac{3}{4} n^2 + \frac{1}{2} n - \frac{1}{4}) d_{-4} \right. \\ &\quad \left. + (\frac{n^4}{24} - \frac{n^3}{3} + \frac{7}{8} n^2 - \frac{5}{6} n + \frac{1}{8}) e_{-5} - . . . . \right) \\ F'' (t-n\omega) &= \frac{1}{\omega^2} \left( b_{-2} - (n-1) c_{-3} + (\frac{n^2}{2} - \frac{3}{2} n + \frac{1}{2}) d_{-4} - (\frac{n^3}{6} - n^2 + \frac{7}{4} n - \frac{5}{6}) e_{-5} + . . \right) \\ F''' (t-n\omega) &= \frac{1}{\omega^3} \left( c_{-3} - (n-\frac{3}{2}) d_{-4} + (\frac{n^2}{2} - 2n + \frac{7}{4}) e_{-5} - . . . . \right) \\ F^{iv} (t-n\omega) &= \frac{1}{\omega^4} \left( d_{-4} - (n-2) e_{-5} + . . . . \right) \\ F^v (t-n\omega) &= \frac{1}{\omega^5} \left( e_{-5} - . . . . \right) \\ . . . . . \end{aligned} \right\} \quad (169)$$

Making  $n = 0$  in (169), we have

$$\left. \begin{aligned} F' (t) &= \frac{1}{\omega} (a_{-1} + \frac{1}{2} b_{-2} + \frac{1}{3} c_{-3} + \frac{1}{4} d_{-4} + \frac{1}{5} e_{-5} + . . . .) \\ F'' (t) &= \frac{1}{\omega^2} (b_{-2} + c_{-3} + \frac{1}{2} d_{-4} + \frac{5}{6} e_{-5} + . . . .) \\ F''' (t) &= \frac{1}{\omega^3} (c_{-3} + \frac{3}{2} d_{-4} + \frac{7}{4} e_{-5} + . . . .) \\ F^{iv} (t) &= \frac{1}{\omega^4} (d_{-4} + 2e_{-5} + . . . .) \\ F^v (t) &= \frac{1}{\omega^5} (e_{-5} + . . . .) \\ . . . . . \end{aligned} \right\} \quad (170)$$

As above, we emphasize the relative importance of the *first* derivative in practice: thus, for brevity, we write the first of equations (169) in the form

$$F' (t-n\omega) = \frac{1}{\omega} (a_{-1} - B' b_{-2} + C' c_{-3} - D' d_{-4} + E' e_{-5} - . . . .) \quad (171)$$

the coefficients  $B'$ ,  $C'$ ,  $D'$ ,  $E'$  being taken from Table IV with the argument  $n$ .

Formulae (169), (170), and (171) are particularly useful in the computation of derivatives at or near the *end* of a series of functions.

Moreover, when the interval  $n$  approaches unity, formulae (169) and (171) are convenient for computing derivatives corresponding to the argument  $t + n\omega$ , since they enable us to proceed *backwards* from the argument  $t + \omega$  with the interval  $1 - n$ . We shall now solve several examples to illustrate these applications.

EXAMPLE I.—From the following ephemeris of the moon’s right-ascension ( $\alpha$ ), compute the *hourly change* in  $\alpha$  at the instant Feb. 3<sup>d</sup> 20<sup>h</sup> 24<sup>m</sup>.

Date 1898	Moon’s R.A. $\alpha$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
	<sup>h</sup> <sup>m</sup> <sup>s</sup>	<sup>m</sup> <sup>s</sup>	<sup>s</sup>	<sup>s</sup>	<sup>s</sup>	<sup>s</sup>
Feb. 1 0	4 49 39.68	+26 21.18				
1 12	5 16 0.86	26 25.99	+ 4.81			
2 0	5 42 26.85	26 24.73	− 1.26	−6.07	+0.08	
2 12	6 8 51.58	26 17.48	7.25	5.99	0.62	+0.54
3 0	6 35 9.06	26 17.48	12.62	5.37	+0.92	+0.30
3 12	7 1 13.92	26 4.86		−4.45		
4 0	7 27 1.71	+25 47.79	−17.07			

Since the assigned unit of time is 1 hour, we have  $\omega = 12$ ; hence, letting  $t = \text{Feb. 4}^{\text{d}} 0^{\text{h}}$ , we find

$$n = \frac{4^{\text{d}} 0^{\text{h}} 0^{\text{m}} - 3^{\text{d}} 20^{\text{h}} 24^{\text{m}}}{12^{\text{h}}} = 0.30$$

which is the interval reckoned *backwards* from  $t = \text{Feb. 4}^{\text{d}} 0^{\text{h}}$ . Denoting the quantity sought by  $\Delta\alpha$ , we then have

$$\Delta\alpha = F'(t - n\omega)$$

We therefore employ the formula (171): thus, taking the requisite differences from the given series, and their coefficients from Table IV, we obtain

		$a_{-1} = +25^{\text{m}} 47.79^{\text{s}}$
$B' = -0.20$	$b_{-2} = -17.07$	$-B'b_{-2} = -3.414$
$C' = +0.07833$	$c_{-3} = -4.45$	$+C'c_{-3} = -0.349$
$D' = -0.03800$	$d_{-4} = +0.92$	$-D'd_{-4} = +0.035$
$E' = +0.02009$	$e_{-5} = +0.30$	$+E'e_{-5} = +0.006$
		$\therefore \omega F'_{-n} = +25^{\text{m}} 44.07^{\text{s}}$

Whence

$$\Delta\alpha = F'_{-n} = 25^{\text{m}} 44^{\text{s}}.07 \div 12 = 2^{\text{m}} 8^{\text{s}}.672.$$

The change in  $\alpha$  for *one minute* ( $\Delta_1\alpha$ ) is simply

$$\Delta_1\alpha = \frac{\Delta\alpha}{60} = 2^{\text{s}}.1445$$

EXAMPLE II.—From the preceding table of moon’s R.A., compute the *hourly variation* in  $\Delta_1\alpha$  for Feb. 3<sup>d</sup> 12<sup>h</sup>; where, as above,  $\Delta_1\alpha$  denotes the change per *minute* in R.A.

Regarding one hour as the unit of time, it is clear that the value of  $F''(t)$  given by (170) is sixty times the quantity sought: the expression for the required variation is therefore  $\frac{1}{60} F''(t)$ , where  $t$  = Feb. 3<sup>d</sup> 12<sup>h</sup>. Accordingly, using the second of (170), we find

Hr. Var. in  $\Delta_1\alpha$ , Feb. 3<sup>d</sup> 12<sup>h</sup>,

$$= \frac{1}{60} \times \frac{1}{(12)^2} (-12.62 - 5.37 + \frac{11}{2} \times 0.62 + \frac{5}{8} \times 0.54) = -0^s.00196$$

EXAMPLE III.—Given the following values of  $F(T) \equiv \log_e T$ : find  $F'(T)$  for  $T = 75$ .

$T$	$F(T) \equiv \log_e T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
45	3.80666					
50	3.91202	+10536				
55	4.00733	9531	−1005	+175		
60	4.09434	8701	830	134	−41	+ 9
65	4.17439	8005	696	102	32	+12
70	4.24850	7411	594	+ 82	−20	
75	4.31749	+ 6899	− 512			

Taking  $t = 75$ , and using the first of (170), we find

$$F'(t) = \frac{10^{-5}}{5} (6899 - \frac{51}{2} \cdot 2 + \frac{8}{3} \cdot 2 - \frac{2}{4} \cdot 0 + \frac{1}{5} \cdot 2) = +0.01334$$

Since  $F'(T) = \frac{1}{T}$ , we observe that the *true mathematical value* of the computed quantity is—

$$F'(t) = \frac{1}{75} = +0.01333\frac{1}{3}$$

EXAMPLE IV.—From the preceding table of natural logarithms, compute  $F''(T)$  for  $T = 67$ .

We let  $t = 70$ , and proceed by the second of (169), observing that

$$n = \frac{70-67}{5} = 0.60$$

Thus we obtain

$C'' = n-1 = -0.40$	$c_{-3} = +102$	$b_{-2} = -0.00594$
$D'' = \frac{n^2}{2} - \frac{3}{2}n + \frac{1}{2} = +0.197$	$d_{-4} = - 32$	$-C''c_{-3} = + 40.8$
$E'' = \frac{n^3}{6} - n^2 + \frac{7}{4}n - \frac{5}{6} = -0.107$	$e_{-5} = + 9$	$+D''d_{-4} = - 6.3$
		$-E''e_{-5} = + 1.0$
		$\therefore \omega^2 F''_{-n} = -0.00558.5$
		$\therefore F''_{-n} = -0.00022.3$

The true value of this quantity is —

$$F''(T) = -\frac{1}{T^2} = -\frac{1}{(67)^2} = -0.00022.27 \dots$$

61. *Derivatives from STIRLING'S Formula.* — When differences both preceding and following the function  $F(t)$  are available, formulae more convenient and accurate than the foregoing may be employed. The most useful and important of these are derived from STIRLING'S Formula of interpolation (104), which is —

$$F_n = F_0 + na + Bb_0 + Cc + Dd_0 + Ee + \dots \quad (172)$$

where the differences are taken according to the schedule on page 62,  $a$ ,  $c$ , and  $e$  being the *mean* differences defined by (101); and where  $B$ ,  $C$ ,  $\dots$  have the values

$$\left. \begin{aligned} B &= \frac{n^2}{2} \\ C &= \frac{n(n^2-1)}{6} = \frac{n^3}{6} - \frac{n}{6} \\ D &= \frac{n^2(n^2-1)}{24} = \frac{n^4}{24} - \frac{n^2}{24} \\ E &= \frac{n(n^2-1)(n^2-4)}{120} = \frac{n^5}{120} - \frac{n^3}{24} + \frac{n}{30} \\ &\dots \end{aligned} \right\} \quad (173)$$

Whence, deriving the values of  $B'$ ,  $B''$ ,  $\dots$ ,  $C'$ ,  $C''$ ,  $\dots$ , etc., from (173), and substituting these (with the above differences) in the general formulae (159), we get

$$\left. \begin{aligned} F' (t+n\omega) &= \frac{1}{\omega} \left( a + nb_0 + \left(\frac{n^2}{2} - \frac{1}{6}\right)c + \left(\frac{n^3}{6} - \frac{n}{12}\right)d_0 + \left(\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{30}\right)e + \dots \right) \\ F'' (t+n\omega) &= \frac{1}{\omega^2} \left( b_0 + nc + \left(\frac{n^2}{2} - \frac{1}{12}\right)d_0 + \left(\frac{n^3}{6} - \frac{n}{4}\right)e + \dots \right) \\ F''' (t+n\omega) &= \frac{1}{\omega^3} \left( c + nd_0 + \left(\frac{n^2}{2} - \frac{1}{4}\right)e + \dots \right) \\ F^{iv} (t+n\omega) &= \frac{1}{\omega^4} \left( d_0 + ne + \dots \right) \\ F^v (t+n\omega) &= \frac{1}{\omega^5} \left( e + \dots \right) \\ &\dots \end{aligned} \right\} \quad (174)$$

Making  $n = 0$  in (174), the latter become

$$\left. \begin{aligned} F' (t) &= \frac{1}{\omega} (a - \frac{1}{6}c + \frac{1}{36}e - \dots) \\ F'' (t) &= \frac{1}{\omega^2} (b_0 - \frac{1}{12}d_0 + \dots) \\ F''' (t) &= \frac{1}{\omega^3} (c - \frac{1}{4}e + \dots) \\ F^{iv} (t) &= \frac{1}{\omega^4} (d_0 - \dots) \\ F^v (t) &= \frac{1}{\omega^5} (e - \dots) \\ &\dots \end{aligned} \right\} \quad (175)$$

Again, writing  $-n$  for  $n$  in (174), we obtain

$$\left. \begin{aligned} F' (t-n\omega) &= \frac{1}{\omega} \left( a - nb_0 + \left(\frac{n^2}{2} - \frac{1}{6}\right)c - \left(\frac{n^3}{6} - \frac{n}{12}\right)d_0 + \left(\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{36}\right)e - \dots \right) \\ F'' (t-n\omega) &= \frac{1}{\omega^2} \left( b_0 - nc + \left(\frac{n^2}{2} - \frac{1}{12}\right)d_0 - \left(\frac{n^3}{6} - \frac{n}{4}\right)e + \dots \right) \\ F''' (t-n\omega) &= \frac{1}{\omega^3} \left( c - nd_0 + \left(\frac{n^2}{2} - \frac{1}{4}\right)e - \dots \right) \\ F^{iv} (t-n\omega) &= \frac{1}{\omega^4} \left( d_0 - ne + \dots \right) \\ F^v (t-n\omega) &= \frac{1}{\omega^5} \left( e - \dots \right) \\ &\dots \end{aligned} \right\} \quad (176)$$

The coefficients for the computation of  $F'(t \pm n\omega)$ , namely —

$$\begin{aligned} B' &= n, & D' &= \frac{n^3}{6} - \frac{n}{12} \\ C' &= \frac{n^2}{2} - \frac{1}{6}, & E' &= \frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{36} \end{aligned} \quad (177)$$

are given in Table V with the argument  $n$ . The quantity  $F'(T)$  is thus readily computed (for any value of  $T$ ) by either one or both of the formulae

$$F' (t+n\omega) = \frac{1}{\omega} (a + nb_0 + C'e + D'd_0 + E'e) \quad (178)$$

$$F' (t-n\omega) = \frac{1}{\omega} (a - nb_0 + C'e - D'd_0 + E'e) \quad (179)$$

in which the *odd* differences are algebraic *means* of the tabular differences, taken as indicated below :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$t - \omega$	$F_{-1}$		$b'$		$d'$	
$t$	$F_0$	$a'$ ( $a$ ) $a_1$	$b_0$	$c'$ ( $c$ ) $c_1$	$d_0$	$e'$ ( $e$ ) $e_1$
$t + \omega$	$F_1$		$b_1$		$d_1$	

The formulae (174) and (175) may also be obtained by the following method, which reverses the preceding order of development by deriving first the *particular*, and from the latter, the more *general* of the two groups in question.

Expanding  $F(t+n\omega)$  by TAYLOR'S Theorem, we have

$$F(t+n\omega) = F(t) + n\omega F'(t) + \frac{n^2\omega^2}{\underline{2}} F''(t) + \frac{n^3\omega^3}{\underline{3}} F'''(t) + \dots \quad (180)$$

Arranging STIRLING'S Formula (104) according to ascending powers of  $n$ , we find

$$\left. \begin{aligned} F(t+n\omega) = & F_0 + n(a - \frac{1}{6}c + \frac{1}{36}e - \dots) + \frac{n^2}{\underline{2}}(b_0 - \frac{1}{12}d_0 + \dots) \\ & + \frac{n^3}{\underline{3}}(c - \frac{1}{4}e + \dots) + \frac{n^4}{\underline{4}}(d_0 - \dots) + \frac{n^5}{\underline{5}}(e - \dots) \\ & + \dots \end{aligned} \right\} \quad (181)$$

Whence, by equating coefficients of like powers of  $n$  in the equivalent expressions (180) and (181), we obtain

$$\left. \begin{aligned} \omega F'(t) &= a - \frac{1}{6}c + \frac{1}{36}e - \dots & \omega^4 F^{iv}(t) &= d_0 - \dots \\ \omega^2 F''(t) &= b_0 - \frac{1}{12}d_0 + \dots & \omega^5 F^v(t) &= e - \dots \\ \omega^3 F'''(t) &= c - \frac{1}{4}e + \dots & & \dots \end{aligned} \right\} \quad (181a)$$

which agree with the formulae (175).

Again, by TAYLOR'S Theorem, we have

$$\begin{aligned} F'(t+n\omega) &= F'(t) + n\omega F''(t) + \frac{n^2\omega^2}{\underline{2}} F'''(t) + \dots \\ F''(t+n\omega) &= F''(t) + n\omega F'''(t) + \frac{n^2\omega^2}{\underline{2}} F^{iv}(t) + \dots \\ &\dots \end{aligned}$$

which may be written in the form

$$\begin{aligned} F'(t+n\omega) &= \frac{1}{\omega} \left( \omega F'(t) + n\omega^2 F''(t) + \frac{n^2}{2} \omega^3 F'''(t) + \dots \right) \\ F''(t+n\omega) &= \frac{1}{\omega^2} \left( \omega^2 F''(t) + n\omega^3 F'''(t) + \frac{n^2}{2} \omega^4 F^{(4)}(t) + \dots \right) \\ &\dots \end{aligned}$$

Substituting in these equations the expressions for  $\omega F'(t)$ ,  $\omega^2 F''(t)$ ,  $\dots$ , as given by (181a), we get

$$\left. \begin{aligned} F'(t+n\omega) &= \frac{1}{\omega} \left[ \left( a - \frac{1}{6}c + \frac{1}{36}e - \dots \right) + n \left( b_0 - \frac{1}{12}d_0 + \dots \right) \right. \\ &\quad \left. + \frac{n^2}{2} \left( c - \frac{1}{4}e + \dots \right) + \frac{n^3}{6} \left( d_0 - \dots \right) + \frac{n^4}{24} \left( e - \dots \right) + \dots \right] \\ F''(t+n\omega) &= \frac{1}{\omega^2} \left[ \left( b_0 - \frac{1}{12}d_0 + \dots \right) + n \left( c - \frac{1}{4}e + \dots \right) + \frac{n^2}{2} \left( d_0 - \dots \right) \right. \\ &\quad \left. + \frac{n^3}{6} \left( e - \dots \right) + \dots \right] \\ F'''(t+n\omega) &= \frac{1}{\omega^3} \left[ \left( c - \frac{1}{4}e + \dots \right) + n \left( d_0 - \dots \right) + \frac{n^2}{2} \left( e - \dots \right) + \dots \right] \\ F^{(4)}(t+n\omega) &= \frac{1}{\omega^4} \left[ \left( d_0 - \dots \right) + n \left( e - \dots \right) + \dots \right] \\ F^{(5)}(t+n\omega) &= \frac{1}{\omega^5} \left[ \left( e - \dots \right) + \dots \right] \\ &\dots \end{aligned} \right\} \quad (182)$$

These expressions, upon being arranged according to the successive orders of differences, will be found identical with the formulae (174). For some purposes, however, the present form is more convenient.

It is quite common, particularly in an astronomical ephemeris, to tabulate the values of  $F'(T)$  corresponding to the tabular values of  $F(T)$ . Such a table would run as follows:\*

$T$	$F(T)$	$F'(T)$
$t - 2\omega$	$F_{-2}$	$F'(t - 2\omega)$
$t - \omega$	$F_{-1}$	$F'(t - \omega)$
$t$	$F_0$	$F'(t)$
$t + \omega$	$F_1$	$F'(t + \omega)$
$t + 2\omega$	$F_2$	$F'(t + 2\omega)$

\* It is evident that  $F'(t+n\omega)$  can be derived from the column  $F'(T)$  by direct interpolation: moreover, when the tabular values of  $F'(T)$  are thus available, this method of computing  $F'(t+n\omega)$  is more expeditious than the use of formula (178).

The first of the formulae (175) is almost invariably used for this purpose, because of its simplicity and rapid convergence; this formula is, in fact, the most important and useful of those which pertain to the computation of derivatives. For this reason we formulate the following

**RULE** for computing the first derivative of a tabular function corresponding to one of the given functional values: *From the mean of the two first differences which immediately precede and follow the function in question, subtract one-sixth ( $\frac{1}{6}$ ) the mean of the corresponding third differences, and divide the result by the tabular interval.* This rule neglects only 5th and higher differences. To include 5th and 6th differences, *add to the above terms (before dividing by  $\omega$ ) one-thirtieth ( $\frac{1}{30}$ ) the mean of the corresponding fifth differences, and divide by  $\omega$  as before.*

It will evidently suffice, in most cases, to apply only the first part of the above rule.

Several examples will now be solved as an exercise in the use of the preceding formulae.

**EXAMPLE I.**—Given the following ephemeris of the sun's declination ( $\delta$ ): compute the *hourly difference* in  $\delta$  for the dates Jan. 7, 10, 13, and 16.

Date 1898	Sun's Decl. $\delta$	$\Delta'$	$\Delta''$	$\Delta'''$	$a$	$-\frac{1}{6}c$	Diff. for 1 hour
	<sup>°</sup> <sup>'</sup> <sup>"</sup>	<sup>'</sup> <sup>"</sup>	<sup>'</sup> <sup>"</sup>	<sup>"</sup>	<sup>"</sup>	<sup>"</sup>	<sup>"</sup>
Jan. 1	−22 59 2.4	+17 23.9					
4	22 41 38.5	21 26.1	+4 2.2				
7	22 20 12.4	25 23.0	3 56.9	−5.3	+1404.55	+0.98	+19.52
10	21 54 49.4	29 13.5	3 50.5	6.4	1638.25	1.12	22.77
13	21 25 35.9	32 56.9	3 43.4	7.1	1865.20	1.27	25.92
16	20 52 39.0	36 32.2	3 35.3	8.1	+2084.55	+1.45	+28.97
19	20 16 6.8		+3 26.0	−9.3			
22	−19 36 8.6	+39 58.2					

The term  $\frac{1}{30}e$  in the first of (175) is here insensible; hence, for each of the given dates we have only to compute the quantity

$$F'(t) = \frac{1}{\omega} (a - \frac{1}{6}c)$$

Accordingly, in column  $a$  we write the required *mean* first differences, expressed in seconds of arc. The next column contains *minus* one-

sixth of the corresponding mean third differences. Finally, since  $\omega = 72$  hours, we write in the last column  $\frac{1}{72}$  of the quantities formed by summing the corresponding terms of the two preceding columns. We thus obtain the hourly differences required.

EXAMPLE II.—Compute, from the ephemeris of the last example, the *daily motion* in declination for the date Jan. 6<sup>d</sup> 13<sup>h</sup> 30<sup>m</sup>.

We proceed *backwards* from Jan. 7, using the formula (179), and taking the coefficients from Table V with the argument

$$n = \frac{7^{\text{d}}\ 0^{\text{h}}\ 0^{\text{m}} - 6^{\text{d}}\ 13^{\text{h}}\ 30^{\text{m}}}{3^{\text{d}}} = \frac{10^{\text{h}}.5}{72^{\text{h}}} = 0.14583$$

Thus we find

			$a = +23^{\circ}\ 24.55''$
$n = 0.14583$	$b_0 = +236.9$	$-nb_0 = -$	$34.55$
$C' = -0.1560$	$c = -5.85$	$+C'c = +$	$0.91$
$D' = -0.012$	$d_0 = -1.1$	$-D'd_0 = -$	$0.01$
			$\therefore \omega F'_{-n} = +22^{\circ}\ 50.90'$

Whence, for the daily motion in  $\delta$ , Jan. 6<sup>d</sup> 13<sup>h</sup> 30<sup>m</sup>, we obtain

$$F'_{-n} = 22^{\circ}\ 50''.90 \div 3 = +7^{\circ}\ 36''.97$$

EXAMPLE III.—The following table gives  $F(T) \equiv e^T$ , where  $e$  denotes the base of natural logarithms: compute  $F'(T)$  for  $T=0.30$ .

$T$	$F(T) \equiv e^T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{\text{iv}}$	$\Delta^{\text{v}}$
0.0	1.000000					
0.1	1.105171	+105171				
0.2	1.221403	116232	+11061			
0.3	1.349859	128456	12224	+1163	+123	+11
0.4	1.491825	141966	13510	1286	134	18
0.5	1.648721	156896	14930	1420	152	+10
0.6	1.822119	173398	16502	1572	+162	
0.7	2.013753	+191634	+18236	+1734		

Using the first of (175), we find

$$F'(0.30) = \frac{10^{-6}}{0.1} \left( 135211 - \frac{1353}{6} + \frac{14.5}{30} \right) = 1.34986$$

It will be observed that our result is substantially equal to the value of  $F(T)$  for the same argument,  $T=0.30$ : this is required by the relation

$$F(T) = F'(T) = F''(T) = \dots = e^T$$

EXAMPLE IV.—From the table of Example III, compute  $F''(T)$  for  $T = 0.462$ .

Taking  $t = 0.4$  and  $n = 0.62$ , we obtain, by means of the second of (174),

		$b_0 = +0.014930$
$n = 0.62$	$c = +1496$	$nc = + 927.5$
$D'' = \frac{n^2}{2} - \frac{1}{12} = +0.1089$	$d_0 = + 152$	$D''d_0 = + 16.6$
$E'' = \frac{n^3}{6} - \frac{n}{4} = -0.115$	$e = + 14$	$E''e = - 1.6$
		$\therefore \omega^2 F_n'' = +0.0158725$
		$\therefore F_n'' = +1.58725$

The true mathematical value is —

$$F''(T) = F(T) = e^T = e^{0.462} = 1.587245 \dots$$

62. *Derivatives from BESSEL'S Formula.*—Other useful formulae, convenient for the computation of tabular derivatives, are those derived from BESSEL'S Formula of interpolation (111). The latter may be written in the form

$$F_n = F_0 + na_1 + Bb + Cc_1 + Dd + Ee_1 + \dots \quad (183)$$

where the differences are taken as in the schedule on page 62,  $b$  and  $d$  being the *mean* differences defined by (106); and where  $B, C, \dots$  have the following values :

$$\left. \begin{aligned} B &= \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \\ C &= \frac{n(n-1)(n-\frac{1}{2})}{6} = \frac{n^3}{6} - \frac{n^2}{4} + \frac{n}{12} \\ D &= \frac{(n+1)n(n-1)(n-2)}{24} = \frac{n^4}{24} - \frac{n^3}{12} + \frac{n^2}{24} + \frac{n}{12} \\ E &= \frac{(n+1)n(n-1)(n-2)(n-\frac{1}{2})}{120} = \frac{n^5}{120} - \frac{n^4}{48} + \frac{n^3}{48} - \frac{n}{120} \\ &\dots \dots \dots \end{aligned} \right\} \quad (184)$$

Deriving from (184) the values of  $B', B'', \dots, C', C'', \dots$ , etc., according to (154), and substituting these in the general formulae (159), we obtain

$$\left. \begin{aligned}
 F' (t+n\omega) &= \frac{1}{\omega} \left( a_1 + (n-\tfrac{1}{2})b + (\tfrac{n^2}{2} - \tfrac{n}{2} + \tfrac{1}{12})c_1 + (\tfrac{n^3}{6} - \tfrac{n^2}{4} - \tfrac{n}{12} + \tfrac{1}{12})d \right. \\
 &\quad \left. + (\tfrac{n^4}{24} - \tfrac{n^3}{12} + \tfrac{n^2}{24} - \tfrac{1}{120})e_1 + \dots \right) \\
 F'' (t+n\omega) &= \frac{1}{\omega^2} \left( b + (n-\tfrac{1}{2})c_1 + (\tfrac{n^2}{2} - \tfrac{n}{2} - \tfrac{1}{12})d + (\tfrac{n^3}{6} - \tfrac{n^2}{4} + \tfrac{1}{24})e_1 + \dots \right) \\
 F''' (t+n\omega) &= \frac{1}{\omega^3} \left( c_1 + (n-\tfrac{1}{2})d + (\tfrac{n^2}{2} - \tfrac{n}{2})e_1 + \dots \right) \\
 F^{\text{iv}} (t+n\omega) &= \frac{1}{\omega^4} \left( d + (n-\tfrac{1}{2})e_1 + \dots \right) \\
 F^{\text{v}} (t+n\omega) &= \frac{1}{\omega^5} \left( e_1 + \dots \right) \\
 &\dots
 \end{aligned} \right\} \quad (185)$$

Putting  $n = 0$  in (185), we get

$$\left. \begin{aligned}
 F' (t) &= \frac{1}{\omega} (a_1 - \tfrac{1}{2}b + \tfrac{1}{12}c_1 + \tfrac{1}{12}d - \tfrac{1}{120}e_1 - \dots) \\
 F'' (t) &= \frac{1}{\omega^2} (b - \tfrac{1}{2}c_1 - \tfrac{1}{12}d + \tfrac{1}{24}e_1 + \dots) \\
 F''' (t) &= \frac{1}{\omega^3} (c_1 - \tfrac{1}{2}d + 0^* + \dots) \\
 F^{\text{iv}} (t) &= \frac{1}{\omega^4} (d - \tfrac{1}{2}e_1 - \dots) \\
 F^{\text{v}} (t) &= \frac{1}{\omega^5} (e_1 - \dots) \\
 &\dots
 \end{aligned} \right\} \quad (186)$$

Again, putting  $n = \frac{1}{2}$  in (185), we obtain the following simple formulae:

$$\left. \begin{aligned}
 F' (t + \tfrac{1}{2}\omega) &= \frac{1}{\omega} (a_1 - \tfrac{1}{24}c_1 + \tfrac{3}{640}e_1 - \dots) \\
 F'' (t + \tfrac{1}{2}\omega) &= \frac{1}{\omega^2} (b - \tfrac{5}{24}d + \dots) \\
 F''' (t + \tfrac{1}{2}\omega) &= \frac{1}{\omega^3} (c_1 - \tfrac{1}{8}e_1 + \dots) \\
 F^{\text{iv}} (t + \tfrac{1}{2}\omega) &= \frac{1}{\omega^4} (d - \dots) \\
 F^{\text{v}} (t + \tfrac{1}{2}\omega) &= \frac{1}{\omega^5} (e_1 - \dots) \\
 &\dots
 \end{aligned} \right\} \quad (187)$$

which determine the derivatives of  $F(T)$  at points *midway* between the tabular values of the function. It is important to observe that,

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\* The coefficient of  $e_1$  vanishes.

unless third differences are considerable, a close *approximation* to  $F'(t + \frac{1}{2}\omega)$  is given by the simple expression

$$F'(t + \frac{1}{2}\omega) = \frac{a_1}{\omega} = \frac{F_1 - F_0}{\omega} \quad (187a)$$

which differs from the *exact* formula only by the omission of the small quantity

$$\frac{1}{\omega} (-\frac{1}{24} \Delta''' + \dots)$$

The formulae for the derivatives of  $F(t - n\omega)$  are deduced from (111a). Let us put, for brevity,

$$\bar{b} = \frac{1}{2}(b_0 + b') \quad , \quad \bar{d} = \frac{1}{2}(d_0 + d') \quad (188)$$

and (111a) becomes

$$F_{-n} = F_0 - na' + B\bar{b} - Cc' + D\bar{d} - Ee' + \dots \quad (189)$$

Comparing this expression with the general formula (160), we find that  $\alpha, \beta, \gamma, \delta, \epsilon, \dots$ , in the latter, are replaced by  $a', \bar{b}, c', \bar{d}, e', \dots$  in (189); hence, observing these changes, and substituting the above determined values of  $B', B'', \dots, C', C'', \dots$ , etc., in the formulae (161), we obtain

$$\left. \begin{aligned} F'(t - n\omega) &= \frac{1}{\omega} \left( a' - (n - \frac{1}{2})\bar{b} + (\frac{n^2}{2} - \frac{n}{2} + \frac{1}{2})c' - (\frac{n^3}{6} - \frac{n^2}{4} - \frac{n}{2} + \frac{1}{2})\bar{d} \right. \\ &\quad \left. + (\frac{n^4}{24} - \frac{n^3}{12} + \frac{n}{24} - \frac{1}{120})e' - \dots \right) \\ F''(t - n\omega) &= \frac{1}{\omega^2} \left( \bar{b} - (n - \frac{1}{2})c' + (\frac{n^2}{2} - \frac{n}{2} - \frac{1}{2})\bar{d} - (\frac{n^3}{6} - \frac{n^2}{4} + \frac{1}{4})e' + \dots \right) \\ F'''(t - n\omega) &= \frac{1}{\omega^3} \left( c' - (n - \frac{1}{2})\bar{d} + (\frac{n^2}{2} - \frac{n}{2})e' - \dots \right) \\ F^{iv}(t - n\omega) &= \frac{1}{\omega^4} \left( \bar{d} - (n - \frac{1}{2})e' + \dots \right) \\ F^v(t - n\omega) &= \frac{1}{\omega^5} \left( e' - \dots \right) \\ &\dots \end{aligned} \right\} \quad (190)$$

The values of  $B', C', D$ , and  $E'$ , as computed from the expressions

$$\left. \begin{aligned} B' &= n - \frac{1}{2} \\ C' &= \frac{n^2}{2} - \frac{n}{2} + \frac{1}{2} \end{aligned} \quad , \quad \left. \begin{aligned} D' &= \frac{n^3}{6} - \frac{n^2}{4} - \frac{n}{2} + \frac{1}{2} \\ E' &= \frac{n^4}{24} - \frac{n^3}{12} + \frac{n}{24} - \frac{1}{120} \end{aligned} \right\} \quad (191)$$

are given in Table VI with the argument  $n$ . By means of these coefficients, values of  $F'(T)$  are readily computed from either one of the formulae

$$F'(t+n\omega) = \frac{1}{\omega}(a_1+B'b+C'e_1+D'd+E'e_1)$$

(192)

$$F'(t-n\omega) = \frac{1}{\omega}(a'-B'\bar{b}+C'e'-D'\bar{d}+E'e')$$

(193)

in which the *even* differences are *means*, taken as indicated below :

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$t-\omega$	$F_{-1}$		$b'$		$d'$	
		$a'$	$(\bar{b})$	$c'$	$(\bar{d})$	$e'$
$t$	$F_0$		$b_0$		$d_0$	
		$a_1$	$(b)$	$c_1$	$(d)$	$e_1$
$t+\omega$	$F_1$		$b_1$		$d_1$	

Several examples will now be solved.

EXAMPLE I. — Given the following table of natural sines :

$T$	$F(T) \equiv \sin T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
40°	0.6427876				
42	0.6691306	+263430			
44	0.6946584	255278	−8152	−312	
46	0.7193398	246814	8464	300	+12
48	0.7431448	238050	8764	−290	+10
50	0.7660444	+228996	−9054		

Let it be required to find  $F'(T)$  for  $T = 45^\circ$ .

Taking  $t = 44^\circ$ , we have

$$\omega = 2^\circ = \frac{\pi}{90} = 0.0349066$$

$$n = \frac{1}{2}$$

Hence, using the first of (187), we find

$$a_1 = +0.0246814$$

$$c_1 = -300; \quad -\frac{1}{24}c_1 = +12.5$$

$$\therefore \omega F'_{\frac{1}{2}} = +0.02468265$$

$$\therefore F'_{\frac{1}{2}} = +0.707106$$

The true value of this quantity is —

$$F'(T) = \cos T = \cos 45^\circ = 0.707107$$

EXAMPLE II.—From the preceding table, compute the value of  $F''(T)$  for  $T = 44^\circ 48'$ .

We take  $t = 44^\circ$ ; hence  $n = 0.40$ . Accordingly, from the second of (185), we obtain

$$\begin{array}{rclclcl}
 & & & & b = -0.0008614 & & \\
 C'' = n - \frac{1}{2} = -0.10 & & c_1 = -300 & & C''c_1 = + & 30 & \\
 D'' = \frac{n^2}{2} - \frac{n}{2} - \frac{1}{12} = -0.203 & & d = + 11 & & D''d = - & 2 & \\
 \hline
 & & & & \therefore \omega^2 F''_n = -0.0008586 & & \\
 & & & & \therefore F''_n = -0.70465 & & 
 \end{array}$$

The actual value is —

$$F''(T) = -\sin T = -\sin 44^\circ 48' = -0.70463$$

EXAMPLE III.—The table below gives the Washington mean time of moon's upper transit at the meridian of Washington:

WASHINGTON MOON CULMINATIONS.

Date 1898	Mean Time of Transit	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
	<sup>h</sup> <sup>m</sup>	<sup>m</sup>	<sup>m</sup>	<sup>m</sup>	<sup>m</sup>
Mar. 22	0 15.57				
23	1 1.00	+45.43	+0.86	+0.44	
24	1 47.29	46.29	1.30	+0.06	-0.38
25	2 34.88	47.59	1.36	-0.30	0.36
26	3 23.83	48.95	1.06	-0.67	-0.37
27	4 13.84	50.01	+0.39		
28	5 4.24	+50.40			

Before proposing an example from this ephemeris, it is proper to remark that the tabular function is the *time* of the moon's arrival at a succession of meridians (in reality one fixed meridian) whose common difference of longitude is 24 hours. The *argument* of the series is therefore the terrestrial *longitude* traversed by the moon, counted west from the Washington meridian: the *interval* of this argument is 24 hours of longitude.

Now, let  $D$  denote the *difference in time of transit for 1 hour of longitude*. This quantity is simply the first derivative of the tabular function: computed for the instant of transit at a meridian  $l$  hours west of Washington, the quantity  $D$  expresses the amount by which the local time of transit at the meridian  $l+1$  hours would exceed the local time of transit at the meridian  $l$  hours, supposing the rate of

retardation to remain constant between the two transits, and equal to what it is at the moment of the first. Thus, if  $D_0$  is the value of  $D$  for the instant of transit at Washington on Mar. 24, the local time of moon's transit at a station 20 minutes west of Washington is given with sufficient precision by the formula

$$\tau = \text{Mar. } 24^{\text{d}} 1^{\text{h}} 47^{\text{m}}.29 + \frac{1}{3} D_0$$

Now, by the first of equations (186), we find for the value of  $D_0$ ,

$$D_0 = F'(t) = \frac{1}{24} \left( 47.59 - \frac{1.33}{2} + \frac{0.06}{12} - \frac{0.37}{12} \right) = 1^{\text{m}}.954$$

Hence the preceding equation gives

$$\tau = \text{Mar. } 24^{\text{d}} 1^{\text{h}} 47^{\text{m}}.94$$

In this manner the local time of transit is simply and accurately determined for any number of stations within half an hour of the Washington meridian.

To find the local time of moon's transit over a meridian 3 hours west of Washington, on the 24th day of March, we have only to interpolate the Washington time of transit between the tabular values for Mar. 24 and Mar. 25, as given above, the interval from the former being

$$n = \frac{3^{\text{h}}}{24^{\text{h}}} = 0.125$$

Finally, if it were required to compute the local time of transit for *several* stations whose longitudes range from  $2\frac{1}{2}$  to  $3\frac{1}{2}$  hours west of Washington, we should find the time for the 3 hour meridian by direct interpolation, as explained above. We should also compute  $D = F'(T)$  for the same meridian; that is, for  $n = 0.125$ . Then the local time of transit at any adjacent meridian, whose longitude from Washington is  $3^{\text{hr}} + \lambda^{\text{min}}$ , is given by the simple formula

$$\tau = \tau_1 + \frac{\lambda}{60} D$$

where  $\tau_1$  is the time of transit at the 3 hour meridian.

EXAMPLE IV — From the preceding ephemeris, compute the *difference in time of transit for 1 hour of longitude* ( $D$ ) at the instant of

moon's transit over the meridian of San Francisco, Mar. 25, 1898; the longitude from Washington being taken as  $3^{\text{h}} 1^{\text{m}} 30^{\text{s}} = 3^{\text{h}}.025$ .

Here we use the formula (192) : thus, taking the coefficients from Table VI (with the argument  $n = 3.025 \div 24 = 0.12604$ ), and the differences from the given ephemeris, we obtain

$$\begin{array}{rcl}
 B' = -0.3740 & b = +1.21 & a_1 = +48.95^{\text{m}} \\
 C' = +0.0282 & c_1 = -0.30 & B'b = -0.453 \\
 D' = +0.0692 & d = -0.365 & C'c_1 = -0.008 \\
 & & D'd = -0.025 \\
 & & \therefore \omega F'_n = +48.464
 \end{array}$$

$$\therefore D = F'_n = 48^{\text{m}}.464 \div 24 = +2^{\text{m}}.019$$

EXAMPLE V.—Use the above table of Moon Culminations to find the variation in  $D$  for 24 hours of longitude, at the instant of moon's *lower* transit over the meridian of Washington, Mar. 24, 1898.

The *lower* transit at Washington is evidently the *upper* transit over the meridian 12 hours west. Hence, denoting the required variation by  $V$ , and regarding 1 hour of longitude as the unit, we find by the second of (187), for  $t = \text{Mar. 24}$ ,

$$\begin{aligned}
 V &= 24F''(t + \tfrac{1}{2}\omega) = \frac{24}{\omega^2} (b - \tfrac{5}{24}d + \dots) \\
 &= \tfrac{1}{24} (1.33 + \tfrac{5}{24} \times 0.37) = +0^{\text{m}}.059
 \end{aligned}$$

63. *Interpolation of Functions by Means of their Tabular First Derivatives.*—As already observed, it frequently happens that a table giving  $F(T)$  also contains the values of  $F'(T)$  which correspond to the tabular functions. The object in thus tabulating the derivative is to facilitate the interpolation of intermediate values of  $F(T)$ . To derive the formula upon which this method is based, we consider the schedule below, where the differences are those of the series  $F'(T)$ :

$T$	$F(T)$	$F'(T)$	1st Diff.	2d	3d
$t - 2\omega$	$F_{-2}$	$F'_{-2}$			
$t - \omega$	$F'_{-1}$	$F'_{-1}$	$\alpha''$	$\beta'$	
$t$	$F_0$	$F'_0$	$\alpha'$	$\beta_0$	$\gamma'$
$t + \omega$	$F_1$	$F'_1$	$\alpha_1$	$\beta_1$	$\gamma_1$
$t + 2\omega$	$F_2$	$F'_2$	$\alpha_2$		

We shall assume that the differences of  $F(T)$  beyond  $\Delta^{\text{iv}}$  may be disregarded; hence the differences of  $F'(T)$  beyond  $\gamma$  may be neglected in the above schedule. Now, by TAYLOR'S Theorem, we have

$$F_n = F_0 + n\omega F_0' + \frac{n^2\omega^2}{1^2} F_0'' + \frac{n^3\omega^3}{1^3} F_0''' + \frac{n^4\omega^4}{1^4} F_0^{\text{iv}} + \dots \quad (194)$$

Again, since

$$F_0'' = \frac{dF'}{dt}, \quad F_0''' = \frac{d^2F'}{dt^2}, \quad F_0^{\text{iv}} = \frac{d^3F'}{dt^3}, \quad \text{etc.},$$

we obtain, by means of the formulae (175),

$$F_0'' = \frac{1}{\omega} (\alpha - \frac{1}{6}\gamma) \quad , \quad F_0''' = \frac{\beta_0}{\omega^2} \quad , \quad F_0^{\text{iv}} = \frac{\gamma}{\omega^3} \quad (195)$$

in which we have put, for brevity,

$$\alpha = \frac{1}{2}(\alpha' + \alpha_1) \quad , \quad \gamma = \frac{1}{2}(\gamma' + \gamma_1) \quad (196)$$

Substituting these expressions for  $F_0''$ ,  $F_0'''$ , and  $F_0^{\text{iv}}$  in (194), the latter becomes

$$F_n = F_0 + n\omega F_0' + \frac{n^2\omega}{1^2} (\alpha - \frac{1}{6}\gamma) + \frac{n^3\omega}{1^3} \beta_0 + \frac{n^4\omega}{1^4} \gamma$$

which may be written

$$F_n = F_0 + n\omega \left( F_0' + \frac{n}{2}\alpha + \frac{n^2}{6}\beta_0 + \frac{n}{1^2}(\frac{n^2}{2}-1)\gamma \right) \quad (197)$$

By means of this formula we compute  $F_n$  in terms of the differences of  $F'(T)$ , instead of the differences of  $F(T)$  direct, as in the usual formulae of interpolation.

Substituting  $-n$  for  $n$  in (197), we have

$$F_{-n} = F_0 - n\omega \left( F_0' - \frac{n}{2}\alpha + \frac{n^2}{6}\beta_0 - \frac{n}{1^2}(\frac{n^2}{2}-1)\gamma \right) \quad (198)$$

The values of

$$B \equiv \frac{n^2}{6} \quad , \quad \Gamma \equiv \frac{n}{1^2}(\frac{n^2}{2}-1) \quad (199)$$

are given in Table VIII with the argument  $n$ . By means of these coefficients we readily compute

$$F_n = F_0 + n\omega (F'_0 + \frac{n}{2}\alpha + B\beta_0 + \Gamma\gamma) \quad (200)$$

$$F_{-n} = F_0 - n\omega (F'_0 - \frac{n}{2}\alpha + B\beta_0 - \Gamma\gamma) \quad (201)$$

The coefficients in Table VIII are not extended beyond  $n=0.60$ , since by this method it is invariably more convenient to proceed from the *nearest* function  $F_0$ .

EXAMPLE.—From the *American Ephemeris* for 1898 we take the heliocentric longitude of *Mercury*, together with the *daily motion* in longitude, for a portion of the month of October. The differences of the daily motion are then taken, as shown below :

Date 1898	Helioc. Long. of <i>Mercury</i>	Daily Motion	$\alpha$	$\beta$	$\gamma$	$\delta$
	° ' "	° ' "	' "	' "	"	"
Oct. 11	176 51 7.8	4 2 34.3				
13	184 41 59.2	3 48 34.3	—14 0.0			
15	192 6 33.3	3 36 16.8	12 17.5	+1 42.5	—5.4	
17	199 8 10.6	3 25 36.4	10 40.4	1 37.1	6.1	—0.7
19	205 49 59.6	3 16 27.0	9 9.4	1 31.0	—7.4	—1.3
21	212 14 54.7	3 8 41.2	—7 45.8	+1 23.6		

Let it be required to find the heliocentric longitude of *Mercury* for the date Oct. 15<sup>d</sup> 14<sup>h</sup> 24<sup>m</sup>.0.

Here we have

$$\begin{aligned} t &= \text{Oct. 15}^{\text{d}} & T &= \text{Oct. 15}^{\text{d}} 14^{\text{h}} 24^{\text{m}}.0 = \text{Oct. 15}^{\text{d}}.60 \\ \omega &= 2^{\text{d}} & n\omega &= T - t = 0^{\text{d}}.60 & n &= 0.30 \end{aligned}$$

Hence, using Table VIII, in connection with (200), we obtain

$F_0 = 192^{\circ} 6' 33.3''$	$\alpha = -11' 28.95''$	$F'_0 = +3^{\circ} 36' 16.8''$
$\frac{n}{2} = +0.15$	$\beta_0 = +1 37.1$	$\frac{n}{2}\alpha = -1 43.34$
$B = +0.0150$	$\gamma = -0 5.75$	$B\beta_0 = +1.46$
$\Gamma = -0.0239$		$\Gamma\gamma = +0.14$
$\therefore \text{Sum, } D = +3 34 35.06$		

Whence

$$F_n = F_0 + n\omega . D = 194^{\circ} 15' 18''.3$$

Differencing the given series of longitudes and applying BESSEL'S Formula of interpolation, we find

$$F_n = 194^\circ 15' 18''.2$$

64. *Application of the Preceding Method of Interpolation when the Second Differences of the Series  $F(T)$  are Nearly Constant.*—When the 3d and 4th differences of  $F(T)$  are small enough to be neglected, we may omit the terms containing  $\beta_0$  and  $\gamma$  in the formulae (197) and (198) : we therefore obtain

$$F_n = F_0 + n\omega (F'_0 + \frac{n}{2}\alpha) \quad (202)$$

$$F_{-n} = F_0 - n\omega (F'_0 - \frac{n}{2}\alpha) \quad (203)$$

It will be interesting to determine the error of these approximate formulae as applied when the 3d differences of  $F(T)$  are appreciable. For this purpose we write (197) in the form

$$F_n = F_0 + n\omega (F'_0 + \frac{n}{2}\alpha) + \frac{n^3}{6}\omega\beta_0 + (\frac{n^4}{24} - \frac{n^2}{12})\omega\gamma$$

Hence, if we disregard 4th differences of  $F(T)$ , and thus neglect  $\gamma$ , it follows that the error in question is —

$$\epsilon = \pm \frac{n^3}{6}\omega\beta_0 \quad (204)$$

Now, from (175), we have

$$F'''(t) = \frac{c}{\omega^3} = \frac{A'''}{\omega^3}$$

also, from (195),

$$F'''(t) = \frac{\beta_0}{\omega^2}$$

Whence

$$\omega\beta_0 = c = A''' \quad (205)$$

and (204) becomes

$$\epsilon = \pm \frac{n^3}{6} A''' \quad (206)$$

Since in practice the maximum value of  $n$  is 0.50, it follows that the maximum error resulting from an application of the formulae (202) and (203), when 3d differences of  $F(T)$  are sensible, is  $\frac{1}{48} A'''$ . Hence, even when third differences are considerable, these formulae are sufficiently accurate for many purposes.

That the formulae (202) and (203) are *rigorously* true when the 3d differences of  $F(T)$  are *zero* may be clearly shown from geometrical considerations, as follows :

The 2d differences of  $F(T)$  being supposed constant, it follows from Theorem VI that the function is necessarily of the form

$$F(T) \equiv a_0 T^2 + a_1 T + a_2 \quad (207)$$

Now, if in the accompanying figure we draw the rectangular co-ordinate axes  $OT$  and  $OY$ , and plot the curve defined analytically by (207) (regarding  $y = F(T)$  as the ordinate corresponding to the abscissa  $T$ ), it is evident that we obtain a *parabola* whose axis is parallel to  $OY$ .

Let us now take

$$OM = t$$

$$OS = t + \omega$$

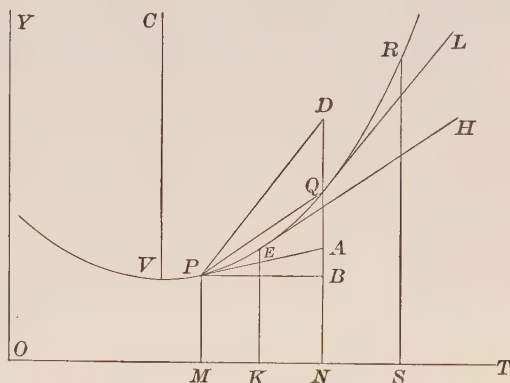
$$ON = t + n\omega$$

## Whence

$$MN = n\omega$$

$$MP = F(t) = F_0$$

$$NQ = F(t+n\omega) = F_n$$



Draw the tangents  $PA, QL$ ; also, draw  $PD \parallel QL$  and  $PB \parallel MN$ .

Then, denoting  $\frac{dF}{dT}$  by  $F'_n$ , we have

$$F_0' = \tan APB$$

$$F'_u = \tan DPB$$

Hence we find

$$NA = MP + PB \tan APB = F_0 + n_{\omega} F_0'$$

$$ND = MP + PB \tan DPB = F_0 + n_{\omega} F_n'$$

It is therefore evident that to find  $NQ = F_n$ , which lies between  $NA$  and  $ND$ , we must employ a value of  $F'$  somewhere between the values  $F'_0$  and  $F''_n$ . Now, let  $KE$  be the ordinate erected at the middle point of  $MN$ , and  $EH$  the tangent at  $E$ . Then, by an elementary

theorem of the parabola, the chord  $PQ$  is *parallel* to  $EH$ , and we have, therefore,

$$F'_n = NQ = MP + PB \tan QPB = F_0 + n\omega F'_{\frac{1}{2}} \quad (208)$$

which agrees with the formula (202).

We have shown above that the maximum error produced by applying this formula when the second differences of  $F(T)$  are *not* constant, is  $\frac{1}{48} \Delta'''$ . Hence, unless the 2d differences of  $F'(T)$  are considerable, we may compute  $F'_n$  by the following

**RULE:** *Find by simple interpolation the value of the tabular derivative which belongs midway between the required function and the nearest tabular function ( $F_0$ ); multiply this quantity ( $F'_{\frac{1}{2}}$ ) by the units contained in the entire interval ( $T-t$ ), and apply the product to  $F_0$ .*

**EXAMPLE I.**—Given the following ephemeris of the moon's declination ( $\delta$ ): compute the value for the date July 9<sup>d</sup> 5<sup>h</sup> 18<sup>m</sup>.0.

Date 1898	Moon's Decl. $\delta$	Diff. for 1 Minute	$\alpha$	$\beta$
July 9 <sup>d</sup> 1 <sup>h</sup>	+ 6 <sup>°</sup> 2' 14.1"	+13.876	"	"
9 4	6 43 39.0	13.732	−0.144	−.006
9 7	7 24 37.4	13.582	0.150	−.010
July 9 10	+8 5 8.0	+13.422	−0.160	

Here  $\omega = 3^h = 180^m$ ; hence, taking  $t = \text{July } 9^d 4^h$ , we find

$$n = \frac{78^m}{180^m} = 0.433 \quad \frac{n}{2} = 0.217$$

Accordingly, the value of  $F''$  interpolated for *half* the interval, or 39 minutes, is—

$$F'_{\frac{n}{2}} = F'_0 + \frac{n}{2} \alpha = 13''.732 - 0.217 \times 0''.147 = 13''.700$$

Whence we obtain

$$\delta = 6^\circ 43' 39''.0 + 78 \times 13''.700 = 7^\circ 1' 27''.6$$

Since the value of  $n$  is nearly one-half, we may interpolate *backwards* from July 9<sup>d</sup> 7<sup>h</sup> with equal facility: thus we find

$$\begin{aligned} n &= 0.567 & \frac{n}{2} &= 0.283 \\ \therefore F'_{-\frac{n}{2}} &= 13''.582 + 0.283 \times 0''.155 = 13''.626 \end{aligned}$$

Whence

$$\delta = 7^{\circ} 24' 37''.4 - 102 \times 13''.626 = 7^{\circ} 1' 27''.55$$

which substantially agrees with the above result.

EXAMPLE II.—From the following table of the moon’s horizontal parallax ( $\pi$ ), interpolate the value for July 10<sup>d</sup> 16<sup>h</sup> 24<sup>m</sup>.0.

Date 1898	Moon’s Hor. Parallax	Diff. for 1 Hour	$\alpha$
July 10. <sup>d</sup> 0	56' 26.1"	−2.04	"
10.5	56' 2.5"	1.89	+0.15
11.0	55' 40.7"	1.73	0.16
11.5	55' 21.1"	−1.55	+0.18

Here we have

$T = \text{July } 10^{\text{d}} 16^{\text{h}}.40$

$t = \text{July } 10^{\text{d}} 12^{\text{h}}.00$

$\omega = 12 \text{ hours}$

$n = \frac{4^{\text{h}}.40}{12^{\text{h}}} = 0.367$

$\frac{n}{2} = 0.183$

We therefore obtain

$F'_{\frac{n}{2}} = -1''.89 + 0.183 \times 0''.16 = -1''.86$

$\therefore \pi = 56' 2''.5 - 4.4 \times 1''.86 = 55' 54''.3$

Interpolating *backwards* from July 11<sup>d</sup> 0<sup>h</sup>, we find

$$\pi = 55' 40''.7 + 7.6 \times 1''.78 = 55' 54''.2$$

65. *Choice of Formulae in a Given Case.*—When derivatives are required to be computed at or near either the *beginning* or the *end* of a tabular series, the formulae derived from NEWTON’S Formula of interpolation must necessarily be employed. In all other cases, the choice lies between STIRLING’S and BESSEL’S forms, and should be decided by the value of  $n$ . When  $n = 0$ , the formulae (175) are unquestionably the best. When  $n = \frac{1}{2}$ , the group (187) is especially convenient. As a general rule, subject to change in certain cases, it may be stated that when  $n$  lies between the limits 0.25 and 0.75, the formulae derived from BESSEL’S Formula of interpolation will be found most convenient: for other values of  $n$ , those derived from STIRLING’S Formula should be employed.

## EXAMPLES.

1. Given the following table of "Latitude Reduction":

$\varphi$	$\varphi - \varphi'$	$\varphi$	$\varphi - \varphi'$
$^{\circ}$	$' \quad ''$	$^{\circ}$	$' \quad ''$
0	0 0.00	15	5 44.32
5	1 59.53	20	7 22.80
10	3 55.47	25	8 47.93

Compute the variation of  $\varphi - \varphi'$  corresponding to a change of  $10'$  in  $\varphi$ , for each of the tabular values of the argument. Denote this variation by  $v$ .

2. From the preceding table, find the change in  $v$  corresponding to a change of one degree in  $\varphi$ , for  $\varphi = 9^{\circ} 30'$ ; also for  $\varphi = 22^{\circ} 42'$ .

3. The table below contains the obliquity of the ecliptic ( $\epsilon$ ) for every fifth century.

Year, A.D.	$\epsilon$
	$^{\circ} \quad ' \quad ''$
0	23 41 43.78
500	37 57.97
1000	34 8.07
1500	30 15.43
2000	23 26 21.41

Compute the variation of  $\epsilon$  per century ( $\epsilon'$ ) for the years 750 and 1250.

4. From the table of  $\epsilon$  in Example 3, find the variation of  $\epsilon'$  per century, for the years 0 and 2000;  $\epsilon'$  denoting the change in  $\epsilon$  for 1 century.

5. Given the logarithm of the earth's radius vector ( $\log R$ ) for the following dates:

Date 1898	log $R$	Date 1898	log $R$
Dec. 15	9.9930137	Dec. 24	9.9927353
18	9.9929025	27	9.9926858
21	9.9928085	30	9.9926619

Compute the hourly change in  $\log R$  for the dates Dec. 18<sup>d</sup> 0<sup>h</sup>, Dec. 22<sup>d</sup> 12<sup>h</sup>, and Dec. 26<sup>d</sup> 17<sup>h</sup>. Denote the hourly change by  $\rho$ .

6. From the preceding ephemeris of  $\log R$ , find the daily variation of  $\rho$  for the dates Dec. 15<sup>d</sup> 0<sup>h</sup>, Dec. 24<sup>d</sup> 0<sup>h</sup>, and Dec. 26<sup>d</sup> 10<sup>h</sup>.

7. The following table gives the right-ascension of *Mercury*, together with the *hourly difference*, for several alternate days of December, 1898 :

Date 1898	R.A. of <i>Mercury</i>	Diff. for 1 Hour
Dec. 1	<sup>h</sup> 18 <sup>m</sup> 1 <sup>s</sup> 2.54	+12.855
3	18 10 50.60	11.587
5	18 19 28.46	9.915
7	18 26 34.57	7.749
9	18 31 43.19	+ 5.009

Compute, by the formulae (200) and (201), the R.A. of *Mercury* for the dates Dec. 4<sup>d</sup> 14<sup>h</sup> 22<sup>m</sup>.0 and Dec. 5<sup>d</sup> 12<sup>h</sup> 30<sup>m</sup>.0. Check the results by direct interpolation from the tabular right-ascensions.

8. Given the following ephemeris of the moon's right-ascension :

Date 1898	Moon's Right-Ascension	Diff. for 1 Minute
Apr. 8 <sup>d</sup> 1	<sup>h</sup> 14 <sup>m</sup> 27 <sup>s</sup> 33.52	2.4508
8 4	14 34 56.35	2.4694
8 7	14 42 22.48	2.4876
8 10	14 49 51.86	2.5054

By the process stated in the rule of §64, compute the moon's R.A. for the dates Apr. 8<sup>d</sup> 3<sup>h</sup> 0<sup>m</sup>; 4<sup>h</sup> 54<sup>m</sup>; 5<sup>h</sup> 30<sup>m</sup>; and Apr. 8<sup>d</sup> 7<sup>h</sup> 36<sup>m</sup>.

## CHAPTER IV.

### OF MECHANICAL QUADRATURE.

66. We have shown in the preceding chapter that when a series of equidistant values of any function are known, it is possible to compute special values of the first and higher derivatives of that function, without regard to its analytical form. We shall now consider the inverse problem, namely: *From a series of tabular values of  $F(T)$ , to find*

$$X = \int_{T'}^{T''} F(T) dT$$

*where the limits  $T'$  and  $T''$  are numerically assigned.*

The solution of this important problem is effected by integrating the expression for  $F(t+n\omega)$ , as given by any one of the several formulae of interpolation, and then giving to  $n$  the limiting values which correspond to  $T'$  and  $T''$ . The method is wholly independent of the analytical form of the function  $F(T)$ . It is therefore of especial advantage and importance in the following cases:

(a) *When the function is analytically unknown.* This is the case with graphical records of continuous observations, so frequently made in physical experiments and tests. As a common example we mention the indicator diagrams of a steam engine. It is usually required to find the area comprised between the "pressure" curve, a fixed base line, and two extreme ordinates. This area may be found, in the generality of cases, by the method proposed.

(b) *When the function is analytically known, but is non-integrable.* Under this head are included the most important applications of the method in question. For example, let it be required to find

$$X = \int_{20^{\circ}}^{52^{\circ}} \frac{dT}{\sqrt{1-e^2 \sin^2 T}}$$

where  $e$  is numerically given. We cannot express the indefinite inte-

gral in finite form. If  $e$  is sufficiently small (say  $e = 0.1$ ), we may expand  $(1 - e^2 \sin^2 T)^{-\frac{1}{2}}$  in a series of ascending powers of  $e^2 \sin^2 T$ , and integrate each term of this expansion separately: a very few terms will then suffice to compute  $X$  as accurately as may be required. If, however, the quantity  $e$  is nearly equal to unity (say  $e = 0.9$ ), this series does not converge with sufficient rapidity for practical use, and hence the method of expansion fails.

On the other hand, given *any* value of  $e$  not exceeding unity, we can readily tabulate  $F(T) \equiv (1 - e^2 \sin^2 T)^{-\frac{1}{2}}$  for a series of values such as  $T = 20^\circ, 24^\circ, 28^\circ, \dots, 52^\circ$ . Having differenced these values of  $F$ , it is then a simple matter to compute  $X$  from the numerical data thus furnished. In the nature of the case, however, the process must, in general, be an approximative one; depending, as does the method of interpolation, upon a limited number of (usually approximate) values of the function in question.

The process by which the definite integral of a function is computed from a series of numerical values of that function, is called *mechanical quadrature*, or *numerical integration*. We proceed to develop the formulae which are commonly employed for this purpose.

67. *Quadrature as Based upon NEWTON'S Formula of Interpolation.*—Suppose that  $i+1$  values of  $F(T)$  have been tabulated and differenced as shown in the schedule below:

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$t$	$F_0$	$\Delta_0'$	$\Delta_0''$	$\Delta_0'''$	$\Delta_0^{iv}$	$\Delta_0^v$
$t + \omega$	$F_1$	$\Delta_1'$	$\Delta_1''$	$\Delta_1'''$	$\Delta_1^{iv}$	$\Delta_1^v$
$t + 2\omega$	$F_2$	$\Delta_2'$	$\Delta_2''$	$\Delta_2'''$	$\Delta_2^{iv}$	$\Delta_2^v$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$t + (i-2)\omega$	$F_{i-2}$	$\Delta_{i-2}'$	$\Delta_{i-2}''$	$\Delta_{i-2}'''$	$\Delta_{i-2}^{iv}$	$\Delta_{i-2}^v$
$t + (i-1)\omega$	$F_{i-1}$	$\Delta_{i-1}'$	$\Delta_{i-1}''$	$\Delta_{i-1}'''$	$\Delta_{i-1}^{iv}$	$\Delta_{i-1}^v$
$t + i\omega$	$F_i$	$\Delta_i'$	$\Delta_i''$	$\Delta_i'''$	$\Delta_i^{iv}$	$\Delta_i^v$

Let it be required to find from this table the value of

$$X = \int_t^{t+i\omega} F(T) dT \quad (209)$$



found directly by integrating the expressions for  $B, C, D, \dots$ , as expanded in (163), and then taking the limits of  $n$  according to (213). But the following indirect method seems preferable, since it adds a significance to the result. Let us put

$$Q = \int (1+y)^n dn = \int (1+ny + By^2 + Cy^3 + Dy^4 + \dots) dn \quad (217)$$

where  $y$  is supposed constant. Then, if we also put

$$Q' = \int_0^1 (1+y)^n dn$$

we shall have

$$Q' = 1 + \frac{1}{2}y + \beta y^2 + \gamma y^3 + \delta y^4 + \epsilon y^5 + \zeta y^6 + \dots \quad (218)$$

the coefficients being those defined in (213).

Again, put

$$(1+y)^n = z \quad (219)$$

that is

$$n \log(1+y) = \log z$$

and we find

$$\log(1+y) \cdot dn = \frac{dz}{z}$$

or

$$z dn = \frac{dz}{\log(1+y)} \quad (220)$$

We therefore obtain

$$Q = \int (1+y)^n dn = \int z dn = \int \frac{dz}{\log(1+y)} = \frac{z}{\log(1+y)} + \text{const.} = \frac{(1+y)^n}{\log(1+y)} + \text{const.}$$

Whence

$$\begin{aligned} Q' &= \int_0^1 (1+y)^n dn = \left[ \frac{(1+y)^n}{\log(1+y)} \right]_{n=0}^{n=1} = \frac{y}{\log(1+y)} \\ &= \frac{y}{y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots} = \left( 1 - \frac{y}{2} + \frac{y^2}{3} - \frac{y^3}{4} + \frac{y^4}{5} - \dots \right)^{-1} \end{aligned}$$

Expanding the last expression by the Binomial Theorem, or by direct division, we obtain

$$Q' = 1 + \frac{1}{2}y - \frac{1}{12}y^2 + \frac{1}{24}y^3 - \frac{13}{720}y^4 + \frac{3}{160}y^5 - \frac{863}{60480}y^6 + \dots \quad (221)$$

Whence, comparing (218) and (221), we find

$$\left. \begin{aligned} \beta &= -\frac{1}{12} & \epsilon &= +\frac{3}{160} \\ \gamma &= +\frac{1}{24} & \zeta &= -\frac{863}{60480} \\ \delta &= -\frac{13}{720} & & \dots \end{aligned} \right\} \quad (222)$$

which are the numerical values of the coefficients of formula (216). It therefore appears that the fundamental *coefficients of quadrature* are those in the expansion of  $[\log(1+y)]^{-1}$ .

Let us now regard the functions  $F_0, F_1, F_2, \dots, F_i$  as first differences of an auxiliary functional series which we shall designate  $'F$ . A schedule containing the new series may be conveniently arranged as follows:

$T$	$'F$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$
$t$	$'F_0$	$F_0$			
$t + \omega$	$'F_1$	$F_1$	$\Delta_0'$		
$t + 2\omega$	$'F_2$	$F_2$	$\Delta_1'$	$\Delta_0''$	
	$'F_3$			$\Delta_1''$	$\Delta_0'''$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$t + (i-1)\omega$	$'F_{i-1}$	$F_{i-1}$	$\Delta'_{i-2}$	$\Delta''_{i-2}$	$\Delta'''_{i-3}$
$t + i\omega$	$'F_i$	$F_i$	$\Delta'_{i-1}$		
	$'F_{i+1}$				

The value of  $'F_0$  is entirely arbitrary. Having assigned a convenient value to this quantity, the remaining terms in the series are readily formed by successive additions, thus:

$$'F_1 = 'F_0 + F_0, \quad 'F_2 = 'F_1 + F_1, \quad \dots, \quad 'F_{i+1} = 'F_i + F_i$$

We shall now put the second member of (216) under a form more convenient for computation. By Theorem I, we have

$$\left. \begin{aligned} \sum_{r=0}^{r=i-1} F_r &\equiv F_0 + F_1 + F_2 + \dots + F_{i-1} = 'F_i - 'F_0 \\ \sum_{r=0}^{r=i-1} \Delta_r' &\equiv \Delta_0' + \Delta_1' + \Delta_2' + \dots + \Delta_{i-1}' = F_i - F_0 \\ \sum_{r=0}^{r=i-1} \Delta_r'' &\equiv \Delta_0'' + \Delta_1'' + \Delta_2'' + \dots + \Delta_{i-1}'' = \Delta_i' - \Delta_0' \\ \sum_{r=0}^{r=i-1} \Delta_r''' &\equiv \Delta_0''' + \Delta_1''' + \Delta_2''' + \dots + \Delta_{i-1}''' = \Delta_i'' - \Delta_0'' \\ &\dots \end{aligned} \right\} \quad (223)$$

and hence (216) becomes

$$\begin{aligned} \int_0^i F(t+n\omega) dn &= ('F_i - 'F_0) + \frac{1}{2}(F_i - F_0) + \beta(\Delta_i' - \Delta_0') \\ &\quad + \gamma(\Delta_i'' - \Delta_0'') + \delta(\Delta_i''' - \Delta_0''') + \epsilon(\Delta_i^{iv} - \Delta_0^{iv}) + \dots \end{aligned} \quad (224)$$

This formula possesses the disadvantage of involving differences  $\Delta_i', \Delta_i'', \Delta_i''', \dots$  which are not furnished by the foregoing schedule. To obviate this difficulty, we proceed as follows:

Put

$$q = {}^iF_i + \tfrac{1}{2}F_i + \beta\Delta_i' + \gamma\Delta_i'' + \delta\Delta_i''' + \epsilon\Delta_i^{iv} + \zeta\Delta_i^v + \dots \quad (225)$$

and (224) may then be written

$$\int_0^i F(t+n\omega) dn = q - ({}^iF_0 + \tfrac{1}{2}F_0 + \beta\Delta_0' + \gamma\Delta_0'' + \delta\Delta_0''' + \dots) \quad (226)$$

Upon giving to  $n$ , in formula (75), the values  $+1, 0, -1, -2, -3, -4, \dots$ , successively, we obtain

$$\left. \begin{aligned} {}^iF_i &= {}^iF_{i+1} - F_i \\ F_i &= F_i \\ \Delta_i' &= \Delta'_{i-1} + \Delta''_{i-2} + \Delta'''_{i-3} + \Delta^{iv}_{i-4} + \Delta^v_{i-5} + \dots \\ \Delta_i'' &= \Delta''_{i-2} + 2\Delta'''_{i-3} + 3\Delta^{iv}_{i-4} + 4\Delta^v_{i-5} + \dots \\ \Delta_i''' &= \Delta'''_{i-3} + 3\Delta^{iv}_{i-4} + 6\Delta^v_{i-5} + \dots \\ \Delta_i^{iv} &= \Delta^{iv}_{i-4} + 4\Delta^v_{i-5} + \dots \\ &\dots \end{aligned} \right\} \quad (227)$$

If these expressions be substituted in (225), we shall have  $q$  in terms of the known tabular differences, and hence obtain the required integral from (226). To avoid the labor of numerical reduction incident to this substitution, we derive the result in the following indirect manner: Put

$$\theta = \frac{1}{\log(1+x)} = x^{-1} + \tfrac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4 + \zeta x^5 + \dots \quad (228)$$

Also, take

$$x = \frac{u}{1-u} \quad (229)$$

and we have

$$\left. \begin{aligned} x^{-1} &= u^{-1}(1-u) = u^{-1} - u^0 \\ x^0 &= u^0 \\ x &= u(1-u)^{-1} = u + u^2 + u^3 + u^4 + u^5 + \dots \\ x^2 &= u^2(1-u)^{-2} = u^2 + 2u^3 + 3u^4 + 4u^5 + \dots \\ x^3 &= u^3(1-u)^{-3} = u^3 + 3u^4 + 6u^5 + \dots \\ x^4 &= u^4(1-u)^{-4} = u^4 + 4u^5 + \dots \\ &\dots \end{aligned} \right\} \quad (230)$$

If now we substitute these expressions for  $x^{-1}, x^0, x, x^2, \dots$  in the second member of (228), we obtain  $\theta$  in terms of  $u^{-1}, u^0, u, u^2, \dots$ . But it will be observed that this operation is identical in algebraic form with the substitution above proposed with respect to (227) and (225); for the  $\theta$  operation involves the quantities

$$\theta; \quad x^{-1}, x^0, x, x^2, x^3, \dots; \quad u^{-1}, u^0, u, u^2, u^3, \dots;$$

while the  $q$  operation involves, in precisely the same algebraic relations, the quantities

$$q; \quad {}^1F_i, F_i, A'_i, A''_i, A'''_i, \dots; \quad {}^1F_{i+1}, F_i, A'_{i-1}, A''_{i-2}, A'''_{i-3}, \dots$$

Hence the result for  $q$  will immediately follow when the result for  $\theta$  has been derived. But we may obtain  $\theta$  as a function of  $u$ , in the form required, more simply than by direct substitution of the expressions (230) in (228). For, by (229), we have

$$1+x = \frac{1}{1-u}$$

whence

$$\log(1+x) = -\log(1-u) \quad (231)$$

Therefore, by (228), we find

$$\theta = \frac{1}{\log(1+x)} = -\frac{1}{\log(1-u)} = u^{-1} - \frac{1}{2}u^0 + \beta u - \gamma u^2 + \delta u^3 - \epsilon u^4 + \xi u^5 - \dots \quad (232)$$

Accordingly, writing  $q$  for  $\theta$ ,  ${}^1F_{i+1}$  for  $u^{-1}$ ,  $F_i$  for  $u^0$ ,  $A'_{i-1}$  for  $u$ , etc., as justified by the preceding reasoning, we obtain

$$q = {}^1F_{i+1} - \frac{1}{2}F_i + \beta A'_{i-1} - \gamma A''_{i-2} + \delta A'''_{i-3} - \epsilon A^{iv}_{i-4} + \xi A^{iv}_{i-5} - \dots \quad (233)$$

Substituting this value of  $q$  in (226), and grouping like terms, we get

$$\begin{aligned} \int_0^i \bar{F}(t+n\omega) dn &= ({}^1F_{i+1} - {}^1F_0) - \frac{1}{2}(F_i + F_0) + \beta(A'_{i-1} - A'_0) \\ &\quad - \gamma(A''_{i-2} + A''_0) + \delta(A'''_{i-3} - A'''_0) - \epsilon(A^{iv}_{i-4} + A^{iv}_0) + \dots \end{aligned} \quad (234)$$

Whence, restoring the values of  $\beta, \gamma, \delta, \dots$ , as given in (222), and applying (211), we have

$$\begin{aligned} \int_t^{t+i\omega} \bar{F}(T) dT &= \omega \int_0^i \bar{F}(t+n\omega) dn \\ &= \omega \left\{ ({}^1F_{i+1} - {}^1F_0) - \frac{1}{2}(F_i + F_0) - \frac{1}{12}(A'_{i-1} - A'_0) - \frac{1}{24}(A''_{i-2} + A''_0) \right. \\ &\quad \left. - \frac{19}{720}(A'''_{i-3} - A'''_0) - \frac{3}{160}(A^{iv}_{i-4} + A^{iv}_0) - \frac{863}{60480}(A^{iv}_{i-5} - A^{iv}_0) - \dots \right\} \end{aligned} \quad (235)$$

When the tabulation of the function extends beyond the value  $F_i$ , it is sometimes more convenient to employ the following formula, easily obtained from (224) :

$$\begin{aligned} \int_t^{t+i\omega} F(T) dT &= \omega \int_0^i F(t+n\omega) dn \\ &= \omega \left\{ (F_i - F_0) + \frac{1}{2} (F_i - F_0) - \frac{1}{12} (\Delta'_i - \Delta'_0) + \frac{1}{24} (\Delta''_i - \Delta''_0) \right. \\ &\quad \left. - \frac{1}{720} (\Delta'''_i - \Delta'''_0) + \frac{1}{180} (\Delta^{iv}_i - \Delta^{iv}_0) - \frac{8}{60480} (\Delta^v_i - \Delta^v_0) + \dots \right\} \quad (236) \end{aligned}$$

We here emphasize the fact that the value of  $F_0$  is wholly arbitrary.

68. As an example in the use of formula (235), let it be required to find\*

$$X = \int_{20^\circ}^{44^\circ} \cos T dT$$

using six places of decimals.

The first consideration concerns the tabular interval to be employed. It is desirable to tabulate as few values of the function as are consistent with a convenient schedule of differences. In all cases the differences should sensibly vanish beyond the third or fourth order. Adopting  $\omega = 4^\circ$  as a suitable interval in the present instance, we obtain the following table of  $F(T) \equiv \cos T$ :

$T$	$F$	$F(T) \equiv \cos T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$20^\circ$	0.000000	0.939693				
24	0.939693	0.913545	-26148			
28	1.853238	0.882948	30597	-4449		
32	2.736186	0.848048	34900	4303	+146	+26
36	3.584234	0.809017	39031	4131	172	17
40	4.393251	0.766044	42973	3942	189	+22
44	5.159295	0.719340	-46704	-3731	+211	
	5.878635					

Taking  $t = 20^\circ$ , and assuming the arbitrary quantity  $F_0 = 0$ , we complete the column  $F$  by successive additions. Whence, by (235), we find

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\* In selecting examples of numerical integration for the present chapter, we have in most cases chosen for  $F(T)$  some simple, *integrable* function, whose tabular values are readily taken or formed from various tables in common use. By such selection we gain in simplicity, while losing little or nothing of generality; and, moreover, from thus knowing *a priori* the true value of the integral sought, we are at once informed as to the final accuracy of each application.

$(i = 6)$		${}^IF_7 - {}^IF_0 = +5.878635$
$F_6 + F_0 = +1.659033$		$- \frac{1}{2} (F_6 + F_0) = -0.829516.5$
$\Delta_5' - \Delta_0' = - 20556$		$- \frac{1}{12} (\Delta_5' - \Delta_0') = + 1713.0$
$\Delta_4'' + \Delta_0'' = - 8180$		$- \frac{1}{24} (\Delta_4'' + \Delta_0'') = + 340.8$
$\Delta_3''' - \Delta_0''' = + 65$		$- \frac{1}{720} (\Delta_3''' - \Delta_0''') = - 1.7$
$\Delta_2^{iv} + \Delta_0^{iv} = + 48$		$- \frac{1}{160} (\Delta_2^{iv} + \Delta_0^{iv}) = - 0.9$
<hr/>		
$\log \Sigma = 0.703392$		$\text{sum, } \Sigma = +5.051170$
$\log \omega = 8.843937$		$\omega = 4^\circ = \frac{\pi}{45}$
$\log X = 9.547329$		$\therefore X = 0.352638$

Since  $\int \cos T dT = \sin T$ , we find for the true value of the definite integral,

$$\begin{aligned} X &= \sin 44^\circ - \sin 20^\circ \\ &= 0.694658 - 0.342020 = 0.352638 \end{aligned}$$

If it be required to compute

$$X = \int_{20^\circ}^{28^\circ} \cos T dT$$

from the foregoing table, formula (236) at once serves the purpose. Thus we obtain

$(i = 2)$		${}^IF_2 - {}^IF_0 = +1.853238$
$F_2 - F_0 = -56745$		$+ \frac{1}{2} (F_2 - F_0) = - 28372.5$
$\Delta_1' - \Delta_0' = - 8752$		$- \frac{1}{12} (\Delta_1' - \Delta_0') = + 729.3$
$\Delta_2'' - \Delta_0'' = + 318$		$+ \frac{1}{24} (\Delta_2'' - \Delta_0'') = + 13.3$
$\Delta_2''' - \Delta_0''' = + 43$		$- \frac{1}{720} (\Delta_2''' - \Delta_0''') = - 1.1$
<hr/>		
		$\Sigma = +1.825607$
		$\therefore X = 0.127451$

Here the true value evidently is —

$$X = \sin 28^\circ - \sin 20^\circ = 0.127451$$

69. *Precepts for Computing the Definite Integral when One or Both Limits Fail to Coincide with some Tabular Value of the Argument T.*—Thus far we have considered the limits of the integral

$$X = \int_{T'}^{T''} F(T) dT$$

to be of the form

$$T' = t + i'\omega, \quad T'' = t + i''\omega$$

where  $i'$  and  $i''$  are integers, and hence  $T'$  and  $T''$  are two particular

values of  $T$  for which  $F(T)$  has been tabulated. We shall now consider the more general problem of finding  $X$  when the limits have the form

$$T' = t + n'\omega \quad , \quad T'' = t + n''\omega$$

where  $n'$  and  $n''$  are non-integers — that is, either proper fractions or mixed numbers.

To illustrate the significance of the problem in question, suppose it were required to find by mechanical quadrature the value of

$$X = \int_{21^{\circ} 13' 37''}^{42^{\circ} 46' 54''} \cos T dT$$

Obviously, it would be impracticable to tabulate the function for a series of equidistant values of  $T$ , of which  $T' = 21^{\circ} 13' 37''$  and  $T'' = 42^{\circ} 46' 54''$  are two particular terms. We may, however, employ the same table as was used in the preceding examples, constructed for  $T = 20^{\circ}, 24^{\circ}, 28^{\circ}, \dots, 44^{\circ}$ , and obtain the required result by *interpolation*. Thus, in the examples just mentioned, we have computed the values of  $X$  from the lower limit  $T' = 20^{\circ}$  to the upper limits  $T'' = 44^{\circ}$  and  $28^{\circ}$ , respectively. In like manner, keeping the lower limit always  $= 20^{\circ}$ , we may find the integral corresponding to each of the following values of the upper limit, viz.:

$$T'' = 20^{\circ}, 24^{\circ}, 28^{\circ}, \dots, 44^{\circ}, \text{ respectively ;}$$

that is, for each of the tabular values of  $T$ . Then, having differenced the resulting values of the integral, we may readily find *by interpolation* the values which correspond to the upper limits  $21^{\circ} 13' 37''$  and  $42^{\circ} 46' 54''$ . Denoting these interpolated values by  $X'$  and  $X''$  respectively, we have

$$X' = \int_{20^{\circ}}^{21^{\circ} 13' 37''} \cos T dT \quad , \quad X'' = \int_{20^{\circ}}^{42^{\circ} 46' 54''} \cos T dT$$

and therefore

$$X = \int_{21^{\circ} 13' 37''}^{42^{\circ} 46' 54''} \cos T dT = X'' - X'$$

We leave the detailed solution of this example to the student as a valuable exercise, exhibiting the spirit of the method employed in problems of this type. The process actually used differs somewhat in

form from the method here explained; but the principle remains the same. We proceed to develop the general formulae.

70. Let us put

$$I_i = \int_0^i F(t+n\omega) \, dn$$

(237)

and

$$\Psi(i) = {}^1F_i + \tfrac{1}{2} F_i + \beta A_i' + \gamma A_i'' + \delta A_i''' + \epsilon A_i^{iv} + \dots$$

(238)

where  $i$  denotes an integer. Then (224) becomes

$$I_i = \Psi(i) - \Psi(0)$$

(239)

Let us now suppose that (239) has been computed for  $i = 0, 1, 2, 3, 4, \dots$ , in succession. Then, from the series of values

$$\left. \begin{aligned} I_0 &= \Psi(0) - \Psi(0) \\ I_1 &= \Psi(1) - \Psi(0) \\ I_2 &= \Psi(2) - \Psi(0) \\ &\dots \end{aligned} \right\}$$

(240)

thus determined, it is evident that any *intermediate* value, say  $I_n$ , can be found by interpolation. To derive a general formula for this purpose, we must express the differences of the series (240). Now, by (238), we have

$$\left. \begin{aligned} \Psi(0) &= {}^1F_0 + \tfrac{1}{2} F_0 + \beta A_0' + \gamma A_0'' + \delta A_0''' + \epsilon A_0^{iv} + \dots \\ \Psi(1) &= {}^1F_1 + \tfrac{1}{2} F_1 + \beta A_1' + \gamma A_1'' + \delta A_1''' + \epsilon A_1^{iv} + \dots \\ \Psi(2) &= {}^1F_2 + \tfrac{1}{2} F_2 + \beta A_2' + \gamma A_2'' + \delta A_2''' + \epsilon A_2^{iv} + \dots \\ &\dots \end{aligned} \right\}$$

(241)

whence, observing the general relation

$$A_{s+1}^{(r)} - A_s^{(r)} = A_s^{(r+1)}$$

we derive the following schedule of differences:

Function	1st Differences	2d Differences	3d Differences
$I_0 = \Psi(0) - \Psi(0)$	$F_0 + \tfrac{1}{2} A_0' + \beta A_0'' + \gamma A_0''' + \dots$	$A_0' + \tfrac{1}{2} A_0'' + \beta A_0''' + \dots$	$A_0'' + \tfrac{1}{2} A_0''' + \dots$
$I_1 = \Psi(1) - \Psi(0)$	$F_1 + \tfrac{1}{2} A_1' + \beta A_1'' + \gamma A_1''' + \dots$	$A_1' + \tfrac{1}{2} A_1'' + \beta A_1''' + \dots$	$A_1'' + \tfrac{1}{2} A_1''' + \dots$
$I_2 = \Psi(2) - \Psi(0)$	$F_2 + \tfrac{1}{2} A_2' + \beta A_2'' + \gamma A_2''' + \dots$	$A_2' + \tfrac{1}{2} A_2'' + \beta A_2''' + \dots$	$A_2'' + \tfrac{1}{2} A_2''' + \dots$
$I_3 = \Psi(3) - \Psi(0)$	$\dots$	$\dots$	$\dots$

Therefore, applying NEWTON'S Formula of interpolation, we have

$$\begin{aligned} I_n &= I_0 + n(1\text{st Diff.}) + B(2\text{d Diff.}) + C(3\text{d Diff.}) + \dots \\ &= \Psi(0) - \Psi(0) + n(F_0 + \tfrac{1}{2} A_0' + \beta A_0'' + \gamma A_0''' + \dots) \\ &\quad + B(A_0' + \tfrac{1}{2} A_0'' + \beta A_0''' + \dots) + C(A_0'' + \tfrac{1}{2} A_0''' + \dots) + D(A_0''' + \dots) + \dots \end{aligned}$$

By transposing the term  $-\Psi(0)$  to the first member, and substituting for  $\Psi(0)$  in the second member the expression given by (241), we find

$$\begin{aligned} I_n + \Psi(0) = & ('F_0 + \tfrac{1}{2}F_0 + \beta A_0' + \gamma A_0'' + \delta A_0''' + \dots) \\ & + n(F_0 + \tfrac{1}{2}A_0' + \beta A_0'' + \gamma A_0''' + \dots) \\ & + B(A_0' + \tfrac{1}{2}A_0'' + \beta A_0''' + \dots) + C(A_0'' + \tfrac{1}{2}A_0''' + \dots) + D(A_0''' + \dots) + \dots \end{aligned}$$

Upon arranging the last expression according to the coefficients 1,  $\frac{1}{2}$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , . . . . , it becomes

$$\begin{aligned} I_n + \Psi(0) = & ('F_0 + nF_0 + BA_0' + CA_0'' + DA_0''' + \dots) \\ & + \tfrac{1}{2}(F_0 + nA_0' + BA_0'' + CA_0''' + \dots) \\ & + \beta(A_0' + nA_0'' + BA_0''' + \dots) \\ & + \gamma(A_0'' + nA_0''' + \dots) \\ & + \delta(A_0''' + \dots) \\ & + \dots \end{aligned}$$

Now, it will be observed that the first polynomial in the second member of this equation is simply the expression for  $'F_n$ ,—the quantity derived from the series  $'F_0, 'F_1, 'F_2, \dots$  by interpolation. Similarly, the remaining parentheses contain the expressions for  $F_n, A_n', A_n'', \dots$ , likewise derived by interpolation from their respective series. We therefore have

$$I_n + \Psi(0) = 'F_n + \tfrac{1}{2}F_n + \beta A_n' + \gamma A_n'' + \delta A_n''' + \dots = \Psi(n) \quad (242)$$

Whence

$$\int_0^n F(t+n\omega) dn = I_n = \Psi(n) - \Psi(0) \quad (243)$$

71. In like manner, if we put

$$\varphi(i) = 'F_{i+1} - \tfrac{1}{2}F_i + \beta A_{i-1}' - \gamma A_{i-2}'' + \delta A_{i-3}''' - \dots \quad (244)$$

then, by (234), we have

$$I_i = \int_0^i F(t+n\omega) dn = \varphi(i) - \Psi(0)$$

Therefore, by interpolation (reasoning precisely as above), we obtain

$$\int_0^n F(t+n\omega) dn = \varphi(n) - \Psi(0) \quad (245)$$

Again, writing  $n'$  for the upper limit  $n$  in (243), and  $n''$  for  $n$  in (245), we get

$$\int_0^{n'} F(t+n\omega) dn = \Psi(n') - \Psi(0) \quad , \quad \int_0^{n''} F(t+n\omega) dn = \varphi(n'') - \Psi(0)$$

the difference of which gives

$$\int_{n'}^{n''} F(t+n\omega) dn = \varphi(n'') - \Psi(n') \quad (246)$$

Upon substituting in equations (243) and (245) the expressions for  $\Psi$  and  $\varphi$  as given by (238) and (244), and restoring the numerical values of  $\beta$ ,  $\gamma$ ,  $\delta$ , . . . . from (222), we obtain

$$\begin{aligned} \int_t^{t+n\omega} F(T) dT &= \omega \int_0^n F(t+n\omega) dn \\ &= \omega \left\{ ({}^I F_n - {}^I F_0) + \frac{1}{2} (F_n - F_0) - \frac{1}{12} (A'_n - A'_0) + \frac{1}{24} (A''_n - A''_0) \right. \\ &\quad \left. - \frac{1}{720} (A'''_n - A'''_0) + \frac{1}{160} (A^{iv}_n - A^{iv}_0) - \frac{8}{60480} (A^v_n - A^v_0) + \dots \right\} \quad (247) \end{aligned}$$

$$\begin{aligned} \int_t^{t+n\omega} F(T) dT &= \omega \int_0^n F(t+n\omega) dn \\ &= \omega \left\{ ({}^I F_{n+1} - {}^I F_0) - \frac{1}{2} (F_n + F_0) - \frac{1}{12} (A'_{n-1} - A'_0) - \frac{1}{24} (A''_{n-2} + A''_0) \right. \\ &\quad \left. - \frac{1}{720} (A'''_{n-3} - A'''_0) - \frac{1}{160} (A^{iv}_{n-4} + A^{iv}_0) - \frac{8}{60480} (A^v_{n-5} - A^v_0) - \dots \right\} \quad (248) \end{aligned}$$

In like manner, we derive from (246),

$$\begin{aligned} \int_{t+n'\omega}^{t+n''\omega} F(T) dT &= \omega \int_{n'}^{n''} F(t+n\omega) dn \\ &= \omega \left\{ ({}^I F_{n'+1} - {}^I F_{n'}) - \frac{1}{2} (F_{n'} + F_{n'}) - \frac{1}{12} (A'_{n'-1} - A'_{n'}) - \frac{1}{24} (A''_{n'-2} + A''_{n'}) \right. \\ &\quad \left. - \frac{1}{720} (A'''_{n'-3} - A'''_{n'}) - \frac{1}{160} (A^{iv}_{n'-4} + A^{iv}_{n'}) - \frac{8}{60480} (A^v_{n'-5} - A^v_{n'}) - \dots \right\} \quad (249) \end{aligned}$$

In these formulae the quantities  $n$ ,  $n'$  and  $n''$  are either proper fractions or mixed numbers; while the value of  ${}^I F_0$  is wholly arbitrary.

It frequently happens that we have to compute

$$X = \int_t^T F(T) dT$$

for several different values of  $T$ ; the lower limit remaining fixed and equal to  $t$ . In such cases it is convenient to determine the arbitrary quantity  ${}^I F_0$ , in (247) and (248), such that the sum of the terms having the subscript *zero* will vanish. Accordingly, we may arrange these formulae as follows :

Take

$${}^1F_0 = -\frac{1}{2}F_0 + \frac{1}{12}\Delta_0' - \frac{1}{24}\Delta_0'' + \frac{19}{720}\Delta_0''' - \frac{3}{160}\Delta_0^{iv} + \frac{863}{60480}\Delta_0^v - \dots$$

Then —

(a) When the upper limit falls near the beginning or middle of the tabular series, find

$$\int_t^{t+n\omega} F(T) dT = \omega \int_0^n F(t+n\omega) dn$$
  
$$= \omega ({}^1F_n + \frac{1}{2}F_n - \frac{1}{12}\Delta_n' + \frac{1}{24}\Delta_n'' - \frac{19}{720}\Delta_n''' + \frac{3}{160}\Delta_n^{iv} - \frac{863}{60480}\Delta_n^v + \dots)$$

(b) When the upper limit falls near the end of the series, find

$$\int_t^{t+n\omega} F(T) dT = \omega \int_0^n F(t+n\omega) dn$$
  
$$= \omega ({}^1F_{n+1} - \frac{1}{2}F_n - \frac{1}{12}\Delta_{n-1}' - \frac{1}{24}\Delta_{n-2}'' - \frac{19}{720}\Delta_{n-3}''' - \frac{3}{160}\Delta_{n-4}^{iv} - \frac{863}{60480}\Delta_{n-5}^v - \dots)$$

EXAMPLE I.—Let it be required to find

$$X = \int_{0.42737}^{0.53054} \frac{10 dT}{\sqrt{T(1-T)}}$$

Here we adopt the interval  $\omega = 0.02$ , and proceed to form a table for  $T = 0.42, 0.44, 0.46, \dots, 0.54$ . Instead of tabulating the *given* function, it is more expedient to tabulate  $\omega$  times this quantity. All differences are thus multiplied by the same factor, and hence the final multiplication by  $\omega$  is avoided. We therefore compute

$$F(T) \equiv 0.02 \times \frac{10}{\sqrt{T(1-T)}} = \frac{0.2}{\sqrt{T(1-T)}}$$

for the values of  $T$  given above. The result is as follows:

$T$	${}^1F$	$F(T) \equiv \frac{0.2}{\sqrt{T(1-T)}}$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
0.42	0.000000	0.405220				
0.44	0.405220	0.402912	−2308	+682		
0.46	0.808132	0.401286	1626	660	−22	+ 8
0.48	1.209418	0.400320	966	646	14	8
0.50	1.609738	0.400000	− 320	640	− 6	+12
0.52	2.009738	0.400320	+ 320	+646	+ 6	
0.54	2.410058	0.401286	+ 966			
	2.811344					

The computation is now readily effected by formula (249). Taking  $t = 0.42$ , we make  $'F_0 = 0$ , and complete the auxiliary series  $'F$ . For the values of  $n'$  and  $n''$ , we have

$$n' = \frac{0.42737 - 0.42}{0.02} = 0.3685$$

$$n'' = \frac{0.53054 - 0.42}{0.02} = 5.5270 = 6 - 0.4730$$

Whence, interpolating by NEWTON'S Formula, we obtain

$'F_{n'} = +0.149636.4$	$'F_{n''+1} = +2.621373.8$
$F_{n'} = +0.404288$	$F_{n''} = +0.400748$
$\Delta'_{n'} = -2054$	$\Delta'_{n''-1} = +659$
$\Delta''_{n'} = +673$	$\Delta''_{n''-2} = +642$
$\Delta'''_{n'} = -19$	$\Delta'''_{n''-3} = 0$

Accordingly, by (249), we find

$F_{n''} + F_{n'} = +0.805036$	$(F_{n''+1} - F_{n'}) = +2.471737.4$
$\Delta'_{n''-1} - \Delta'_{n'} = +2713$	$-\frac{1}{2} (F_{n''} + F_{n'}) = -0.402518.0$
$\Delta''_{n''-2} + \Delta''_{n'} = +1315$	$-\frac{1}{1\frac{1}{2}} (\Delta'_{n''-1} - \Delta'_{n'}) = -226.1$
$\Delta'''_{n''-3} - \Delta'''_{n'} = +19$	$-\frac{1}{2\frac{1}{4}} (\Delta''_{n''-2} + \Delta''_{n'}) = -54.8$
	$-\frac{1\frac{9}{20}}{7\frac{9}{20}} (\Delta'''_{n''-3} - \Delta'''_{n'}) = -0.5$
	$\therefore X = +2.068938$

To verify this result, we observe that

$$\int \frac{dT}{\sqrt{T(1-T)}} = 2 \sin^{-1} \sqrt{T}$$

and therefore

$$X = 20 (\sin^{-1} \sqrt{0.53054} - \sin^{-1} \sqrt{0.42737})$$

$$= 20 (168303''.25 - 146965''.80) \sin 1'' = 2.068938$$

EXAMPLE II. — Let it be required to evaluate, by mechanical quadratures, the integrals

$$X_1 = \int_2^{3.2} 60 T^3 dT \quad ; \quad X_2 = \int_2^{4.8} 60 T^3 dT \quad ; \quad \text{and} \quad X_3 = \int_2^{11.6} 60 T^3 dT$$

Here we tabulate  $\omega$  times the given function for  $T = 2, 4, 6, 8, 10, 12$ ; thus we obtain the following table of  $F(T) \equiv 120 T^3$ :

$T$	$'F$	$F(T) \equiv 120T^3$	$\Delta'$	$\Delta''$	$\Delta'''$
2	— 248	960			
4	+ 712	7680	+ 6720	+11520	+5760
6	8392	25920	18240	17280	5760
8	34312	61440	35520	23040	+5760
10	95752	120000	58560	+28800	
12	215752	207360	+87360		
	+423112				

The several values of  $X$  here required are conveniently computed by the formulae (250). Thus (assuming  $t=2$ ) the first step is to determine  $'F_0$ , the computation of which is as follows :

$$\begin{array}{rcl}
 F_0 & = & + 960 \\
 \Delta'_0 & = & + 6720 \\
 \Delta''_0 & = & +11520 \\
 \Delta'''_0 & = & + 5760 \\
 \hline
 & & - \frac{1}{2} F_0 = -480 \\
 & & + \frac{1}{12} \Delta'_0 = +560 \\
 & & - \frac{1}{24} \Delta''_0 = -480 \\
 & & + \frac{19}{720} \Delta'''_0 = +152 \\
 \hline
 & & \therefore 'F_0 = -248
 \end{array}$$

The column  $'F$  is now completed by successive additions of the functions  $F$ , as shown in the table above.

(1) To find  $X_1$ : Here we have

$$n = \frac{3.2-2}{2} = 0.60$$

With this value of  $n$  we readily find  $'F_n$ ,  $F_n$ ,  $\Delta'_n$ ,  $\Delta''_n$  and  $\Delta'''_n$  by interpolation, employing NEWTON'S Formula; whence  $X_1$  is computed by formula (a) of (250). The results are given below :

$$\begin{array}{rcl}
 & & 'F_n = - 26.816 \\
 F_n & = & + 3932.16 \\
 \Delta'_n & = & +12940.8 \\
 \Delta''_n & = & +14976.0 \\
 \Delta'''_n & = & + 5760.0 \\
 \hline
 & & + \frac{1}{2} F_n = +1966.080 \\
 & & - \frac{1}{12} \Delta'_n = -1078.400 \\
 & & + \frac{1}{24} \Delta''_n = + 624.000 \\
 & & - \frac{19}{720} \Delta'''_n = - 152.000 \\
 \hline
 & & \therefore X_1 = +1332.864
 \end{array}$$

All of the quantities above are mathematically exact, and hence the result may be rigorously compared with the known value of the integral: thus, since

$$\int 60T^3dT = 15T^4$$

we have

$$X_1 = 15(3.2^4-2^4) = 1332.864$$

which is identical with the foregoing result.

(2) To find  $X_2$ : We use the same formula as before, the value of  $n$  in this case being

$$n = \frac{4.8-2}{2} = 1.40$$

or an interval of 0.40 counted forward from the quantities  $'F_1$ ,  $F_1$ ,  $\Delta'_1$ ,  $\Delta''_1$ , and  $\Delta'''_1$ . Accordingly we find

	$'F_n = +2461.504$
$F_n = +13271.04$	$+ \frac{1}{2} F_n = +6635.520$
$\Delta'_n = +24460.8$	$- \frac{1}{12} \Delta'_n = -2038.400$
$\Delta''_n = +19584.0$	$+ \frac{1}{24} \Delta''_n = + 816.000$
$\Delta'''_n = + 5760.0$	$- \frac{19}{720} \Delta'''_n = - 152.000$
	$\therefore X_2 = +7722.624$

This result is also mathematically exact, as may be easily verified.

(3) To find  $X_3$ : Since here the upper limit falls near the end of the given series, we employ formula (b) of (250). In this instance the value of  $n$  is—

$$n = \frac{11.6-2}{2} = 4.80 = 5 - 0.20$$

which corresponds to an interval of 0.20 counted *backwards* from the quantities having the subscript *five*. We therefore obtain

$n+1 = 6 -0.20$	$\dots\dots\dots$	$'F_{n+1} = +373075.264$
$n = 5 -0.20$	$F_n = +187307.52$	$- \frac{1}{2} F_n = - 93653.760$
$n-1 = 4 -0.20$	$\Delta'_{n-1} = + 81139.2$	$- \frac{1}{12} \Delta'_{n-1} = - 6761.600$
$n-2 = 3 -0.20$	$\Delta''_{n-2} = + 27648.0$	$- \frac{1}{24} \Delta''_{n-2} = - 1152.000$
$n-3 = 2 -0.20$	$\Delta'''_{n-3} = + 5760.0$	$- \frac{19}{720} \Delta'''_{n-3} = - 152.000$
		$\therefore X_3 = +271355.904$

which is mathematically exact.

72. *Quadrature as Based upon STIRLING'S Formula of Interpolation.*—The preceding formulae of quadrature obviously involve the same disadvantages as are inherent in NEWTON'S Formula of interpolation. We now proceed to integrate the expression for  $F(t+nw)$  as given by STIRLING'S Formula, thus obtaining more convenient and accurate formulae than those already derived. For this purpose, let

the schedule of functions (including  $'F$ ) and differences be taken as below :

$T$	$'F$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$
$t - 2\omega$		$F_{-2}$		$\Delta'_{-2}$	$\Delta'''_{-2}$
$t - \omega$		$F_{-1}$	$\Delta'_{-\frac{1}{2}}$	$\Delta'_{-1}$	$\Delta'''_{-1}$
$t$	$'F_{-\frac{1}{2}}$	$F_0$	$\Delta'_{-\frac{1}{2}}$	$\Delta'_0$	$\Delta'''_0$
$t + \omega$	$'F_{\frac{1}{2}}$	$F_1$	$\Delta'_{\frac{1}{2}}$	$\Delta'_1$	$\Delta'''_1$
$t + 2\omega$	$'F_{\frac{3}{2}}$	$F_2$	$\Delta'_{\frac{3}{2}}$	$\Delta'_2$	$\Delta'''_2$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$t + (i-1)\omega$	.	$F_{i-1}$	.	$\Delta'_{i-1}$	.
$t + i\omega$	$'F_{i-\frac{1}{2}}$	$F_i$	$\Delta'_{i-\frac{1}{2}}$	$\Delta'_i$	$\Delta'''_{i-\frac{1}{2}}$
$t + (i+1)\omega$	$'F_{i+\frac{1}{2}}$	$F_{i+1}$	$\Delta'_{i+\frac{1}{2}}$	$\Delta'_{i+1}$	$\Delta'''_{i+\frac{1}{2}}$
$t + (i+2)\omega$		$F_{i+2}$	$\Delta'_{i+\frac{3}{2}}$	$\Delta'_{i+2}$	$\Delta'''_{i+\frac{3}{2}}$

Reverting now to (104), an inspection of this equation shows clearly the law of formation of the successive coefficients in the second member : hence, adding the term in  $\Delta^{vi}$ , we have

$$\begin{aligned}
 F(t+n\omega) = & F_0 + n \left( \frac{\Delta'_{-\frac{1}{2}} + \Delta'_{\frac{1}{2}}}{2} \right) + \frac{n^2}{2} \Delta''_0 + \frac{n(n^2-1)}{6} \left( \frac{\Delta'''_{-\frac{1}{2}} + \Delta'''_{\frac{1}{2}}}{2} \right) + \frac{n^2(n^2-1)}{24} \Delta^{iv}_0 \\
 & + \frac{n(n^2-1)(n^2-4)}{120} \left( \frac{\Delta^v_{-\frac{1}{2}} + \Delta^v_{\frac{1}{2}}}{2} \right) + \frac{n^2(n^2-1)(n^2-4)}{720} \Delta^{vi}_0 + \dots \dots \dots \quad (251)
 \end{aligned}$$

Multiplying by  $dn$  and integrating, we obtain

$$\begin{aligned}
 \int F(t+n\omega) dn = & nF_0 + \frac{n^2}{4} (\Delta'_{-\frac{1}{2}} + \Delta'_{\frac{1}{2}}) + \frac{n^3}{6} \Delta''_0 + \frac{1}{24} (n^4 - n^2) (\Delta'''_{-\frac{1}{2}} + \Delta'''_{\frac{1}{2}}) \\
 & + \frac{1}{24} (n^5 - n^3) \Delta^{iv}_0 + \frac{1}{240} (n^6 - \frac{5}{4} n^4 + 2n^2) (\Delta^v_{-\frac{1}{2}} + \Delta^v_{\frac{1}{2}}) \\
 & + \frac{1}{720} (n^7 - n^5 + \frac{4}{3} n^3) \Delta^{vi}_0 + \dots \dots \dots + M \quad (252)
 \end{aligned}$$

$M$  being the constant of integration. If this integral is now taken between the limits  $n = -\frac{1}{2}$  and  $n = +\frac{1}{2}$ , the coefficients of  $\Delta'$ ,  $\Delta'''$ ,  $\Delta^v$ ,  $\dots \dots \dots$  evidently vanish, and we find, therefore,

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} F(t+n\omega) dn = F_0 + \frac{1}{24} \Delta''_0 - \frac{1}{5760} \Delta^{iv}_0 + \frac{367}{967680} \Delta^{vi}_0 - \dots \dots \dots \quad (253)$$

In like manner, we derive

$$\left. \begin{aligned} \int_{\frac{1}{2}}^{\frac{3}{2}} F(t+n\omega) dn &= F_1 + \frac{1}{2^4} A_1'' - \frac{17}{5 \cdot 7 \cdot 60} A_1^{iv} + \frac{367}{96 \cdot 7 \cdot 680} A_1^{vi} - . . . \\ . . . . . \\ \int_{-\frac{i-1}{2}}^{-\frac{i+1}{2}} F(t+n\omega) dn &= F_i + \frac{1}{2^4} A_i'' - \frac{17}{5 \cdot 7 \cdot 60} A_i^{iv} + \frac{367}{96 \cdot 7 \cdot 680} A_i^{vi} - . . . \end{aligned} \right\} \quad (254)$$

Whence, by summation, we obtain

$$\int_{-\frac{1}{2}}^{i+\frac{1}{2}} F(t+n\omega) dn = \sum_{r=0}^{r=i} F_r + \frac{1}{24} \sum_{r=0}^{r=i} A_r'' - \frac{1}{576} \frac{1}{768} \sum_{r=0}^{r=i} A_r^{iv} + \frac{1}{96} \frac{3}{6780} \sum_{r=0}^{r=i} A_r^{vi} - \dots \quad (255)$$

Upon substituting the relations

$$\left. \begin{aligned} \sum_{r=0}^{r=i} F_r &= F_0 + F_1 + F_2 + \dots + F_i = {}^I F_{i+\frac{1}{2}} - {}^I F_{-\frac{1}{2}} \\ \sum_{r=0}^{r=i} A_r^{II} &= A_0^{II} + A_1^{II} + A_2^{II} + \dots + A_i^{II} = A_{i+\frac{1}{2}}^{II} - A_{-\frac{1}{2}}^{II} \\ \sum_{r=0}^{r=i} A_r^{iV} &= A_0^{iV} + A_1^{iV} + A_2^{iV} + \dots + A_i^{iV} = A_{i+\frac{1}{2}}^{iV} - A_{-\frac{1}{2}}^{iV} \end{aligned} \right\} \quad (256)$$

in formula (255), the latter becomes

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn = ({}^I F_{+\frac{1}{2}} - {}^I F_{-\frac{1}{2}}) + \frac{1}{2^{\frac{1}{2}}} ({}^I \mathcal{F}'_{+\frac{1}{2}} - {}^I \mathcal{F}'_{-\frac{1}{2}}) - \frac{1}{5^{\frac{1}{2}} 7^{\frac{1}{2}} 6^{\frac{1}{2}}} ({}^I \mathcal{F}'''_{+\frac{1}{2}} - {}^I \mathcal{F}'''_{-\frac{1}{2}}) + \frac{1}{9^{\frac{1}{2}} 6^{\frac{1}{2}} 7^{\frac{1}{2}} 6^{\frac{1}{2}} 8^{\frac{1}{2}} 6^{\frac{1}{2}}} ({}^I \mathcal{F}^{\text{v}}_{+\frac{1}{2}} - {}^I \mathcal{F}^{\text{v}}_{-\frac{1}{2}}) - \dots \quad (257)$$

Finally, therefore, we obtain

$$\int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT = \omega \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn$$

$$= \omega \left\{ {}^I F_{\frac{1}{2}} - {}^I F_{-\frac{1}{2}} + \frac{1}{24} ({}^I A_{\frac{1}{2}} - {}^I A_{-\frac{1}{2}}) - \frac{1}{5760} ({}^I I_{\frac{1}{2}} - {}^I I_{-\frac{1}{2}}) + \frac{367}{967680} ({}^I \nabla_{\frac{1}{2}} - {}^I \nabla_{-\frac{1}{2}}) - \dots \right\} \quad (258)$$

When several values of an integral are to be computed from a given series, each having the lower limit  $t - \frac{1}{2}\omega$ , it will be more convenient and expeditious to determine the arbitrary quantity  $'F_{-\frac{1}{2}}$  such that the sum of the terms with subscript  $-\frac{1}{2}$  is equal to *zero*. The formula (258) may therefore be written as below :

$$\left. \begin{aligned} {}^I F_{-\frac{1}{2}} &= -\frac{1}{24} A'_{-\frac{1}{2}} + \frac{17}{5760} A'''_{-\frac{1}{2}} - \frac{367}{967680} A^V_{-\frac{1}{2}} + \dots \\ \int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT &= \omega \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn \\ &= \omega ({}^I F_{+\frac{1}{2}} + \frac{1}{24} A'_{+\frac{1}{2}} - \frac{17}{5760} A'''_{+\frac{1}{2}} + \frac{367}{967680} A^V_{+\frac{1}{2}} - \dots) \end{aligned} \right\} \quad (259)$$

As an application of (258), let it be required to find

$$X = \int_{30^{\circ}}^{45^{\circ}} \sec^2 T dT$$

Taking  $\omega = 3^{\circ}$ ,  $t = 31^{\circ} 30'$ , and  $'F_{-\frac{1}{2}} = 0$ , we tabulate  $F(T) \equiv \sec^2 T$  as follows :

$T$	$'F$	$F(T) \equiv \sec^2 T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
25 30		1.22751				
28 30		1.29480	+ 6729	+1343		
<b>31 30</b>	<b>0.00000</b>	1.37552	<b>8072</b>	1612	+ <b>269</b>	+ 78
34 30	1.37552	1.47236	9684	1959	347	117
37 30	2.84788	1.58879	11643	2423	464	155
40 30	4.43667	1.72945	14066	3042	619	223
<b>43 30</b>	6.16612	1.90053	17108	3884	842	+326
46 30	<b>8.06665</b>	2.11045	<b>20992</b>	+5052	+ <b>1168</b>	
49 30		2.37089	+26044			

Owing to the rapid convergence of the coefficients in (258), the effect of fifth differences is here insensible : hence, using but three terms of this formula, we obtain

$(i = 4)$   
 $\Delta'_{4\frac{1}{2}} - \Delta'_{-\frac{1}{2}} = +12920$   
 $\Delta'''_{4\frac{1}{2}} - \Delta'''_{-\frac{1}{2}} = + 899$   

---

 $\log \Sigma = 0.906982$   
 $\log \omega = 8.718999$   
 $\log X = 9.625981$

$'F_{4\frac{1}{2}} - 'F_{-\frac{1}{2}} = +8.06665$   
 $+ \frac{1}{24} (\Delta'_{4\frac{1}{2}} - \Delta'_{-\frac{1}{2}}) = + 538.3$   
 $- \frac{17}{5760} (\Delta'''_{4\frac{1}{2}} - \Delta'''_{-\frac{1}{2}}) = - 2.7$   

---

 $\Sigma = +8.07201$   
 $\omega = 3^{\circ} = \pi \div 60$   
 $\therefore X = 0.422650$

*Verification:* Since

$$\int \sec^2 T dT = \tan T$$

we have

$$X = \tan 45^{\circ} - \tan 30^{\circ} = 1 - \frac{1}{3}\sqrt{3} = 0.422650$$

To illustrate the application of formula (259) when several values are assigned in succession to the integer  $i$ , we solve below an example which proceeds according to the evident relation

$$l = l_0 + \int_{T_0}^T \left( \frac{dl}{dT} \right) dT$$

where  $l$  denotes the value of any coördinate at the instant  $T$ , and  $l_0$  its value at the epoch  $T_0$ . In particular, let us put

- $l$  = the heliocentric longitude of *Mars* for any date  $T$ ;
- $\frac{dl}{dT}$  = the daily motion in longitude;
- $T_0$  = 1898 June 13, Greenwich mean noon;
- $l_0 = 1^\circ 47' 14''.3$  = the heliocentric longitude for the date  $T_0$ ;

and let it be required to compute the longitude ( $l$ ) for Greenwich mean noon of the dates

(1898) June 21, June 29, July 7, July 15, and July 23;

the values of the daily motion being taken from the *American Ephemeris* for 1898.

The complete solution is conveniently arranged in tabular form as follows :

Date 1898	$F(T)=8\left(\frac{dl}{dT}\right)$	$\Delta'$	$\Delta''$	$T$	$l_0+{}^1F$	$+ \frac{\Delta'}{24}$	$l$
	$^\circ \quad ' \quad ''$	$''$	$''$		$^\circ \quad ' \quad ''$	$''$	$^\circ \quad ' \quad ''$
June 1	5 1 36.8	−105.5					
9	4 59 51.3	125.9	−20.4				
17	4 57 45.4	144.4	18.5	June 13	1 47 19.5	−5.2	1 47 14
25	4 55 21.0	160.8	16.4	21	6 45 4.9	6.0	6 44 59
July 3	4 52 40.2	175.0	14.2	29	11 40 25.9	6.7	11 40 19
11	4 49 45.2	187.5	12.5	July 7	16 33 6.1	7.3	16 32 59
19	4 46 37.7	198.1	10.6	15	21 22 51.3	7.8	21 22 44
27	4 43 19.6	−206.3	− 8.2	23	26 9 29.0	−8.3	26 9 21
Aug. 4	4 39 53.3						

The function tabulated in column  $F(T)$  is *eight times* the daily motion in  $l$ : it is so multiplied, because the unit of the derivative being one day, we have  $\omega = 8$ ; and thus the final multiplication by this factor is avoided.

Upon taking  $t = \text{June 17}$ , the formula (259) is at once applicable. We have, therefore, since differences beyond  $\Delta''$  are negligible,

$${}^1F_{-\frac{1}{2}} = -\frac{1}{24} \Delta'_{-\frac{1}{2}} = \frac{-125''.9}{-24} = +5''.2$$

and

$$l - l_0 = \int_{T_0 = t - \frac{1}{2} \omega}^{T = t + (t + \frac{1}{2}) \omega} \left(\frac{dl}{dT}\right) dT = {}^1F_{t+\frac{1}{2}} + \frac{1}{24} \Delta'_{t+\frac{1}{2}}$$

the factor  $\omega$  having been previously applied. Whence the expression for  $l$  becomes

$$l = l_0 + {}^1F_{i+\frac{1}{2}} + \frac{1}{24} A'_{i+\frac{1}{2}}$$

Thus, the value of  $l$  for *any* date  $T$  being found by adding the constant  $l_0$  to the integral taken from  $T_0$  to  $T$ , it is clear that we have merely to increase the above value of  ${}^1F_{-\frac{1}{2}}$  by the quantity  $l_0 = 1^\circ 47' 14''.3$  in order to avoid the subsequent addition of this constant to each computed value of the integral. Accordingly, under the heading  $l_0 + {}^1F$ , on the line  $t - \frac{1}{2}\omega (= \text{June } 13)$ , we write the quantity  $1^\circ 47' 19''.5$ ; the remaining numbers of this column are then formed in the usual manner by successive additions of the functions  $F$ . Each term of the series thus formed is evidently greater by  $l_0$  than if the latter constant had been excluded from the initial term.

Under  $+\frac{1}{24}A'$  are written the values derived from the corresponding terms in  $A'$ . The sum  $l_0 + {}^1F + \frac{1}{24}A'$  is then tabulated in the final column,  $l$ , which therefore gives the heliocentric longitude of *Mars* for the dates indicated in column  $T$ .

73. *Applications in which the Limits Fall Otherwise than Midway Between Tabular Values of the Argument and Function.*—If we put

$$\theta(i + \tfrac{1}{2}) = {}^1F_{i+\frac{1}{2}} + \frac{1}{24} A'_{i+\frac{1}{2}} - \frac{1}{8160} A''_{i+\frac{1}{2}} + \frac{367}{987840} A'''_{i+\frac{1}{2}} - \dots \quad (260)$$

the formula (257) becomes

$$\int_{-\frac{1}{2}}^{i+\frac{1}{2}} F(t+n\omega) dn = \theta(i + \tfrac{1}{2}) - \theta(-\tfrac{1}{2}) \quad (261)$$

Whence, if as before  $n$  denotes a fractional or mixed number, we derive, by the general method of interpolation employed in §70,

$$\int_{-\frac{1}{2}}^n F(t+n\omega) dn = \theta(n) - \theta(-\tfrac{1}{2}) \quad (262)$$

Upon substituting  $n'$  and  $n''$  successively for  $n$  in (262), and taking the difference of the resulting expressions, we get

$$\int_{n'}^{n''} F(t+n\omega) dn = \theta(n'') - \theta(n') \quad (263)$$

Finally, replacing the functions  $\theta$ , in (262) and (263), according to the expression (260), we obtain the following formulæ :

$$\int_{t-\frac{1}{2}\omega}^{t+n\omega} F(T) dT = \omega \int_{-\frac{1}{2}}^n F(t+n\omega) dn$$
$$= \omega \left\{ ({}^tF_n - {}^tF_{-\frac{1}{2}}) + \frac{1}{24} (\Delta'_n - \Delta'_{-\frac{1}{2}}) - \frac{1}{5760} (\Delta'''_n - \Delta'''_{-\frac{1}{2}}) + \frac{3}{967680} (\Delta^v_n - \Delta^v_{-\frac{1}{2}}) - \dots \right\} \quad (264)$$

$$\int_{t+n'\omega}^{t+n''\omega} F(T) dT = \omega \int_{n'}^{n''} F(t+n\omega) dn$$
$$= \omega \left\{ ({}^tF_{n''} - {}^tF_{n'}) + \frac{1}{24} (\Delta'_{n''} - \Delta'_{n'}) - \frac{1}{5760} (\Delta'''_{n''} - \Delta'''_{n'}) + \frac{3}{967680} (\Delta^v_{n''} - \Delta^v_{n'}) - \dots \right\} \quad (265)$$

where the quantity  ${}^tF_{-\frac{1}{2}}$  is wholly arbitrary; and where  ${}^tF_n, \Delta'_n, \Delta'''_n, \Delta^v_n, \dots$  (and the similar terms with subscripts  $n'$  and  $n''$ ) are to be found by interpolation.

When several values of an integral are to be computed from a given series by (264), the latter may be modified to the more expedient form given below :

$$\left. \begin{aligned} {}^tF_{-\frac{1}{2}} &= -\frac{1}{24} \Delta'_{-\frac{1}{2}} + \frac{1}{5760} \Delta'''_{-\frac{1}{2}} - \frac{3}{967680} \Delta^v_{-\frac{1}{2}} + \dots \\ \int_{t-\frac{1}{2}\omega}^{t+n\omega} F(T) dT &= \omega \int_{-\frac{1}{2}}^n F(t+n\omega) dn \\ &= \omega ({}^tF_n + \frac{1}{24} \Delta'_n - \frac{1}{5760} \Delta'''_n + \frac{3}{967680} \Delta^v_n - \dots) \end{aligned} \right\} \quad (266)$$

EXAMPLE.—Find the value of

$$X = \int_{0.15}^{0.48} e^x dx$$

$e$  being the base of the natural system of logarithms.

Taking  $\omega = 0.1, t = 0.2$ , and  $F(T) \equiv e^T$ , we prepare the following table :

$T$	${}^tF$	$F(T) \equiv e^T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
0.0		1.000000				
0.1		1.105171	+105171			
0.2	−0.004840	1.221403	116232	+11061		
0.3	+1.216563	1.349859	128456	12224	+1163	+123
0.4	2.566422	1.491825	141966	13510	1286	134
0.5	4.058247	1.648721	156896	14930	1420	152
0.6	+5.706968	1.822119	173398	16502	1572	+162
0.7		2.013753	+191634	+18236	+1734	

Proceeding by formula (266), we find

$${}^tF_{-\frac{1}{2}} = 10^{-6} \left( -\frac{1}{24} \times 116232 + \frac{1}{5760} \times 1163 \right) = -0.004840$$

whence the column  $'F$  is completed as shown above. Denoting the assigned lower and upper limits by  $T'$  and  $T''$ , respectively, we have

$$\begin{aligned} T' &= 0.15 = 0.20 - 0.05 = t - \frac{1}{2}\omega \\ T'' &= 0.48 = 0.20 + 0.28 = t + 2.8\omega \end{aligned}$$

Hence, at the upper limit, the value of  $n$  is —

$$n = 2.8 = 2.5 + 0.30$$

Accordingly, we find  $'F_n$ ,  $A'_n$ , and  $A'''_n$  by interpolating forward from the quantities  $'F_{2.5}$ ,  $A'_{2.5}$ , and  $A'''_{2.5}$  with the interval 0.30. From the table above, we take

$$'F_{2.5} = +4.058247 \quad A'_{2.5} = +0.156896 \quad A'''_{2.5} = +0.001572$$

Hence, making the required interpolations by means of BESSEL'S Formula, and proceeding according to (266), we find

$$\begin{array}{rcl} & & 'F_n = +4.535670.3 \\ A'_n = +0.161674 & + \frac{1}{24} A'_n = + & 6736.4 \\ A'''_n = + 1619 & - \frac{17}{5760} A'''_n = - & 4.8 \\ \hline & \Sigma = & +4.542402 \\ \therefore X = & +0.4542402 & \end{array}$$

The true mathematical value of  $X$  is easily found: thus, since

$$\int e^x dx = e^x$$

we have

$$X = e^{0.48} - e^{0.15} = 0.454240159 \dots$$

74. *Quadrature as Based upon BESSEL'S Formula of Interpolation.*—Another set of formulae for mechanical quadrature, similar to those already developed, may be derived in the same manner from BESSEL'S expression for  $F(t+n\omega)$ . However, since these formulae may be obtained more conveniently by a direct transformation of those developed in the preceding section, we choose the latter course.

Putting  $n'' = i$ , and  $n' = 0$ , in formula (263), we have

$$\int_0^i F(t+n\omega) dn = \theta(i) - \theta(0) \quad (267)$$

We also have, by (260),

$$\theta(i) = 'F_i + \frac{1}{24} A'_i - \frac{17}{5760} A'''_i + \frac{367}{967680} A^{(5)}_i - \dots \quad (268)$$

Referring now to the general schedule on page 147, it will be observed that the quantities

$${}^iF_i, A'_i, A''_i, A''_i, \dots$$

are not explicitly given, but must be found by interpolating *to halves* between  ${}^iF_{i-\frac{1}{2}}$  and  ${}^iF_{i+\frac{1}{2}}$ ,  $A'_{i-\frac{1}{2}}$  and  $A'_{i+\frac{1}{2}}$ , etc., respectively. For this purpose, let us denote the algebraic *means* of the latter pairs of quantities by  $({}^iF_i)$ ,  $(A'_i)$ ,  $(A''_i)$ , . . . ; that is, let us put

$$\left. \begin{aligned} ({}^iF_i) &= \frac{1}{2} ({}^iF_{i-\frac{1}{2}} + {}^iF_{i+\frac{1}{2}}) \\ (A'_i) &= \frac{1}{2} (A'_{i-\frac{1}{2}} + A'_{i+\frac{1}{2}}) \\ (A''_i) &= \frac{1}{2} (A''_{i-\frac{1}{2}} + A''_{i+\frac{1}{2}}) \\ &\dots \end{aligned} \right\} \quad (269)$$

Applying formula (126), we have, therefore,

$$\left. \begin{aligned} {}^iF_i &= ({}^iF_i) - \frac{1}{8} (A'_i) + \frac{3}{128} (A''_i) - \frac{5}{1024} (A''_i) + \dots \\ A'_i &= (A'_i) - \frac{1}{8} (A''_i) + \frac{3}{128} (A''_i) - \dots \\ A''_i &= (A''_i) - \frac{1}{8} (A''_i) + \dots \\ A''_i &= (A''_i) - \dots \\ &\dots \end{aligned} \right\} \quad (270)$$

Upon substituting these values of  ${}^iF_i$ ,  $A'_i$ ,  $A''_i$ , . . . in (268), and reducing, we get

$$\theta(i) = ({}^iF_i) - \frac{1}{12} (A'_i) + \frac{1}{720} (A''_i) - \frac{1}{60480} (A''_i) + \dots \quad (271)$$

Putting  $i = 0$ , this becomes

$$\theta(0) = ({}^0F_0) - \frac{1}{12} (A'_0) + \frac{1}{720} (A''_0) - \frac{1}{60480} (A''_0) + \dots \quad (272)$$

Whence, from (267), we derive

$$\begin{aligned} \int_0^i {}^iF(t+n\omega) dn &= \theta(i) - \theta(0) \\ &= [({}^iF_i) - ({}^0F_0)] - \frac{1}{12} [(A'_i) - (A'_0)] \\ &\quad + \frac{1}{720} [(A''_i) - (A''_0)] - \frac{1}{60480} [(A''_i) - (A''_0)] + \dots \end{aligned} \quad (273)$$

\* It is evident from (111) that the coefficient for the *sixth difference* in BESSEL's Formula is—

$$\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{16}$$

which, for  $n = \frac{1}{2}$ , yields the value given in the text.

Again, putting  $n = i$  in (262), we have

$$\begin{aligned} \int_{-\frac{1}{2}}^i F(t+n\omega) dn &= \theta(i) - \theta(-\tfrac{1}{2}) \\ &= ({}^iF_i) - \tfrac{1}{12}({}^iA'_i) + \tfrac{1}{720}({}^iA''') - \tfrac{1}{60480}({}^iA^v) + \dots \\ &\quad - {}^iF_{-\frac{1}{2}} - \tfrac{1}{24}A'_{-\frac{1}{2}} + \tfrac{1}{5760}A'''_{-\frac{1}{2}} - \tfrac{1}{967680}A^v_{-\frac{1}{2}} + \dots \end{aligned} \quad (274)$$

In like manner, making  $n'' = i + \frac{1}{2}$ , and  $n' = 0$ , in (263), we obtain

$$\begin{aligned} \int_0^{i+\frac{1}{2}} F(t+n\omega) dn &= \theta(i + \tfrac{1}{2}) - \theta(0) \\ &= {}^iF_{i+\frac{1}{2}} + \tfrac{1}{24}A'_{i+\frac{1}{2}} - \tfrac{1}{5760}A'''_{i+\frac{1}{2}} + \tfrac{1}{967680}A^v_{i+\frac{1}{2}} - \dots \\ &\quad - ({}^iF_0) + \tfrac{1}{12}({}^iA'_0) - \tfrac{1}{720}({}^iA''') + \tfrac{1}{60480}({}^iA^v) - \dots \end{aligned} \quad (275)$$

Finally, substituting  $n'' = n$  and  $n' = 0$ , in (263), the latter becomes

$$\begin{aligned} \int_0^n F(t+n\omega) dn &= \theta(n) - \theta(0) \\ &= {}^iF_n + \tfrac{1}{24}A'_n - \tfrac{1}{5760}A'''_n + \tfrac{1}{967680}A^v_n - \dots \\ &\quad - ({}^iF_0) + \tfrac{1}{12}({}^iA'_0) - \tfrac{1}{720}({}^iA''') + \tfrac{1}{60480}({}^iA^v) - \dots \end{aligned} \quad (276)$$

The equations (273), (274), (275) and (276) give, respectively, the following formulae of quadrature :

$$\begin{aligned} \int_t^{t+i\omega} F(T) dT &= \omega \int_0^i F(t+n\omega) dn \\ &= \omega \{ [({}^iF_i) - ({}^iF_0)] - \tfrac{1}{12}[({}^iA'_i) - ({}^iA'_0)] + \tfrac{1}{720}[({}^iA'''_i) - ({}^iA'''_0)] \\ &\quad - \tfrac{1}{60480}[({}^iA^v_i) - ({}^iA^v_0)] + \dots \} \end{aligned} \quad (277)$$

$$\begin{aligned} \int_{t-\frac{1}{2}}^{t+i\omega} F(T) dT &= \omega \int_{-\frac{1}{2}}^i F(t+n\omega) dn \\ &= \omega \{ ({}^iF_i) - \tfrac{1}{12}({}^iA'_i) + \tfrac{1}{720}({}^iA''') - \tfrac{1}{60480}({}^iA^v) + \dots \\ &\quad - {}^iF_{-\frac{1}{2}} - \tfrac{1}{24}A'_{-\frac{1}{2}} + \tfrac{1}{5760}A'''_{-\frac{1}{2}} - \tfrac{1}{967680}A^v_{-\frac{1}{2}} + \dots \} \end{aligned} \quad (278)$$

$$\begin{aligned} \int_t^{t+i\omega+\frac{1}{2}\omega} F(T) dT &= \omega \int_0^{i+\frac{1}{2}} F(t+n\omega) dn \\ &= \omega \{ {}^iF_{i+\frac{1}{2}} + \tfrac{1}{24}A'_{i+\frac{1}{2}} - \tfrac{1}{5760}A'''_{i+\frac{1}{2}} + \tfrac{1}{967680}A^v_{i+\frac{1}{2}} - \dots \\ &\quad - ({}^iF_0) + \tfrac{1}{12}({}^iA'_0) - \tfrac{1}{720}({}^iA''') + \tfrac{1}{60480}({}^iA^v) - \dots \} \end{aligned} \quad (279)$$

$$\begin{aligned} \int_t^{t+n\omega} F(T) dT &= \omega \int_0^n F(t+n\omega) dn \\ &= \omega \{ {}^iF_n + \tfrac{1}{24}A'_n - \tfrac{1}{5760}A'''_n + \tfrac{1}{967680}A^v_n - \dots \\ &\quad - ({}^iF_0) + \tfrac{1}{12}({}^iA'_0) - \tfrac{1}{720}({}^iA''') + \tfrac{1}{60480}({}^iA^v) - \dots \} \end{aligned} \quad (280)$$

in which  $i$  denotes an integer and  $n$  a non-integer; where  $'F_{-\frac{1}{2}}$  is wholly arbitrary; and where  $(F_i)$ ,  $(A'_i)$ , . . . . and  $(F_0)$ ,  $(A'_0)$ , . . . . are *means* of corresponding tabular quantities, as defined by (269).

If, in the formulae (277), (279), and (280), we take

$$('F_0) = \frac{1}{12} (A'_0) - \frac{1}{720} (A''_0) + \frac{1}{60480} (A'''_0) - \dots$$

then the sum of the terms with subscript *zero* will vanish. But, since

$$('F_0) = 'F_{-\frac{1}{2}} + \frac{1}{2} F_0$$

the preceding condition is evidently satisfied if we take

$$'F_{-\frac{1}{2}} = -\frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A''_0) + \frac{1}{60480} (A'''_0) - \dots \quad (281)$$

The formulae (277), (278), (279) and (280) may therefore be computed as follows :

$$\left. \begin{aligned} 'F_{-\frac{1}{2}} &= -\frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A''_0) + \frac{1}{60480} (A'''_0) - \dots \\ \int_t^{t+\omega} F(T) dT &= \omega \int_0^i F(t+n\omega) dn \\ &= \omega \{ ('F_i) - \frac{1}{12} (A'_i) + \frac{1}{720} (A''_i) - \frac{1}{60480} (A'''_i) + \dots \} \end{aligned} \right\} \quad (282)$$

$$\left. \begin{aligned} 'F_{-\frac{1}{2}} &= -\frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{1}{967680} A^{(v)}_{-\frac{1}{2}} + \dots \\ \int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT &= \omega \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn \\ &= \omega \{ ('F_i) - \frac{1}{12} (A'_i) + \frac{1}{720} (A''_i) - \frac{1}{60480} (A'''_i) + \dots \} \end{aligned} \right\} \quad (283)$$

$$\left. \begin{aligned} 'F_{-\frac{1}{2}} &= -\frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A''_0) + \frac{1}{60480} (A'''_0) - \dots \\ \int_t^{t+\omega+\frac{1}{2}\omega} F(T) dT &= \omega \int_0^{i+\frac{1}{2}} F(t+n\omega) dn \\ &= \omega ( 'F_{i+\frac{1}{2}} + \frac{1}{24} A'_{i+\frac{1}{2}} - \frac{1}{5760} A'''_{i+\frac{1}{2}} + \frac{1}{967680} A^{(v)}_{i+\frac{1}{2}} - \dots ) \end{aligned} \right\} \quad (284)$$

$$\left. \begin{aligned} 'F_{-\frac{1}{2}} &= -\frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A''_0) + \frac{1}{60480} (A'''_0) - \dots \\ \int_t^{t+n\omega} F(T) dT &= \omega \int_0^n F(t+n\omega) dn \\ &= \omega ( 'F_n + \frac{1}{24} A'_n - \frac{1}{5760} A'''_n + \frac{1}{967680} A^{(v)}_n - \dots ) \end{aligned} \right\} \quad (285)$$

Several examples will now be solved as an exercise in the use of the preceding formulae.

EXAMPLE I. — Let it be required to find

$$X = \int_0^{\frac{\pi}{2}} T \sin T dT$$

Here we take  $\omega = 20^\circ = \frac{\pi}{9}$ ,  $t = 10^\circ = \frac{\pi}{18}$ , and tabulate  $F(T) \equiv \omega T \sin T$ , as follows :

$T$	$'F$	$F(T) \equiv \omega T \sin T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
$-50^\circ$		+0.23335					
30		0.09139	-14196				
-10		0.01058	-8081	+6115	+1966	-1966	
+10	0.00000	0.01058	0	8081	0	1966	0
30	0.01058	0.09139	+8081	8081	-1966	1966	+361
50	0.10197	0.23335	14196	6115	3571	1605	648
70	0.33532	0.40075	16740	+2544	4528	957	856
90	0.73607	0.54831	14756	-1984	4629	-101	897
110	1.28438	0.62974	+8143	6613	3833	+796	+808
130		0.60671	-2303	10446	-2229	+1604	
+150		+0.45693	-14978	-12675			

The value of  $X$  is now readily found by (278). Taking the arbitrary quantity  $'F_{-\frac{1}{2}} = 0$ , we complete the column  $'F$  as above : we then have

$$'F_{-\frac{1}{2}} = \Delta'_{-\frac{1}{2}} = \Delta'''_{-\frac{1}{2}} = \Delta^v_{-\frac{1}{2}} = 0$$

Whence, proceeding by (278), we find

$$\begin{array}{ll}
 (i = 4) & ('F_4) = \frac{1}{2}('F_{3\frac{1}{2}} + 'F_{4\frac{1}{2}}) = +1.01022.5 \\
 (\Delta'_4) = \frac{1}{2}(\Delta'_{3\frac{1}{2}} + \Delta'_{4\frac{1}{2}}) = +0.11449.5 & -\frac{1}{12}(\Delta'_4) = -954.1 \\
 (\Delta'''_4) = \frac{1}{2}(\Delta'''_{3\frac{1}{2}} + \Delta'''_{4\frac{1}{2}}) = -4231 & +\frac{1}{720}(\Delta'''_4) = -64.6 \\
 (\Delta^v_4) = \frac{1}{2}(\Delta^v_{3\frac{1}{2}} + \Delta^v_{4\frac{1}{2}}) = +852 & -\frac{1}{60480}(\Delta^v_4) = -2.7 \\
 \therefore X = +1.00001
 \end{array}$$

*Verification :* Since

$$\int T \sin T dT = \sin T - T \cos T$$

we have

$$X = \left[ \sin T - T \cos T \right]_0^{\frac{\pi}{2}} = 1$$

EXAMPLE II. — Compute the value of

$$X = \int_{0.9}^{1.2} \frac{dT}{(1 + 0.1 T^2)^{\frac{3}{2}}}$$

Here we take  $\omega = 0.1$ ,  $t = 0.9$ , and tabulate  $F(T) \equiv (1 + 0.1 T^2)^{-\frac{2}{3}}$ , as below :

$T$	$'F$	$F(T) \equiv (1 + 0.1 T^2)^{-\frac{2}{3}}$	$\Delta'$	$\Delta''$	$\Delta'''$
0.7		0.93076	—1961		
0.8		0.91115	2141	—180	
0.9	—0.44672	0.88974	2296	155	+25
1.0	+0.44302	0.86678	2424	128	27
1.1	1.30980	0.84254	2528	104	24
1.2	2.15234	0.81726	2607	79	25
1.3	+2.96960	0.79119	—2664	—57	+22
1.4		0.76455			

Proceeding by means of (282), we compute  $'F_{-\frac{1}{2}}$  as follows :

$$\begin{array}{rclcl} F_0 & = & +0.88974 & -\frac{1}{2} F_0 & = -0.44487 \\ (\Delta'_0) & = & -2218 & +\frac{1}{12} (\Delta'_0) & = -184.8 \\ (\Delta'''_0) & = & +26 & -\frac{11}{720} (\Delta'''_0) & = -0.4 \end{array}$$

$$\therefore 'F_{-\frac{1}{2}} = -0.44672$$

Whence, having completed the column  $'F$ , we conclude the computation by (282), with the following results :

$$\begin{array}{rclcl} (i = 3) & & ('F_3) & = & +2.56097 \\ (\Delta'_3) & = & -0.02567.5 & -\frac{1}{12} (\Delta'_3) & = +214.0 \\ (\Delta'''_3) & = & +23.5 & +\frac{11}{720} (\Delta'''_3) & = +0.4 \end{array}$$

$$\Sigma = +2.56311$$
$$\therefore X = +0.256311$$

Since

$$\int \frac{dT}{(1 + 0.1 T^2)^{\frac{2}{3}}} = \frac{T}{(1 + 0.1 T^2)^{\frac{1}{3}}}$$

we find for the true value of  $X$ ,

$$X = 1.121936 - 0.865625 = 0.256311$$

EXAMPLE III.—Let it be required to find

$$X = \int_{\frac{\pi}{4}}^{\tan^{-1} \frac{3}{2}} \sec^4 T dT$$

Expressing the assigned limits in degrees of arc, they become

$$\frac{\pi}{4} = 45^\circ \qquad \tan^{-1} \frac{3}{2} = 56^\circ 18' 35''.77 = 56^\circ.30994$$

We now take  $\omega = 2^\circ = \frac{\pi}{90}$ ,  $t = 45^\circ$ , and tabulate the following values of  $F(T) \equiv \omega \sec^4 T$ :

$T$	$'F$	$F(T) \equiv \omega \sec^4 T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
41		0.10759					
43		0.12201	+ 1442	+ 320			
45	-0.06819	0.13963	1762	410	+ 90	+ 35	+ 11
47	+0.07144	0.16135	2172	535	125	46	20
49	0.23279	0.18842	2707	706	171	66	38
51	0.42121	0.22255	3413	943	237	104	51
53	0.64376	0.26611	4356	1284	341	155	86
55	0.90987	0.32251	5640	1780	496	241	+146
57	1.23238	0.39671	7420	2517	737	+387	
59	+1.62909	0.49608	9937	+3641	+1124		
61		0.63186	+13578				

Here we employ formula (285); in which, for the upper limit, we have

$$n = (56^\circ.30994 - 45^\circ) \div 2^\circ = 5.65497 = 5.5 + 0.15497$$

For the value of  $'F_{-\frac{1}{2}}$ , we find

$$\begin{array}{rcl} F_0 & = & +0.13963 \\ (\Delta'_0) & = & + 1967 \\ (\Delta''_0) & = & + 108 \end{array} \quad \begin{array}{rcl} - \frac{1}{2} F_0 & = & -0.06981.5 \\ + \frac{1}{2} (\Delta'_0) & = & + 163.9 \\ - \frac{1}{2} (\Delta''_0) & = & - 1.6 \end{array}$$


---


$$\therefore 'F_{-\frac{1}{2}} = -0.06819$$

Whence, completing the series  $'F$ , and observing that the values of  $'F_n$ ,  $\Delta'_n$ , and  $\Delta''_n$  are obtained from their respective series by interpolation with the interval 0.15497, we find

$$\begin{array}{rcl} 'F_n & = & +1.28846.8 \\ \Delta'_n & = & +0.07754 \\ \Delta''_n & = & + 787 \end{array} \quad \begin{array}{rcl} + \frac{1}{24} \Delta'_n & = & + 323.1 \\ - \frac{17}{5760} \Delta''_n & = & - 2.3 \end{array}$$


---


$$\therefore X = +1.29168$$

*Verification:* The expression for the indefinite integral is—

$$\int \sec^4 T dT = \tan T + \frac{1}{3} \tan^3 T$$

Therefore

$$X = \left\{ \frac{3}{2} + \frac{1}{3} \left( \frac{3}{2} \right)^3 \right\} - \left\{ 1 + \frac{1}{3} \right\} = 1.29167$$

with which the above result substantially agrees.

## DOUBLE INTEGRATION BY QUADRATURES.

75. Having derived various formulae for the mechanical quadrature of single integrals, the corresponding formulae for *double integration* are now readily deduced. These will serve to compute integrals of the form

$$Y = \int_{T'} \int_{T''} F(T) dT^2 \quad (286)$$

independently of the analytical nature of the function  $F(T)$ , provided  $T'$  and  $T''$  are numerically assigned. To define the quantity  $Y$  more explicitly, let us put

$$\int F(T) dT = f(T) + M \quad (286a)$$

where  $M$  is the constant of integration. We then have

$$Y = \int_{T'} f(T) dT + M(T'' - T') \quad (287)$$

It is therefore evident that unless the constant  $M$  has a definite value in any given case, the value of  $Y$  will be indeterminate. In practical applications, however, the quantity  $M$  is generally known from the fact that the *first integral* has an assigned value (usually *zero*) corresponding to the lower limit of integration.

If we now put

$$T = t + n\omega \quad , \quad T' = t + n'\omega \quad , \quad T'' = t + n''\omega$$

we have

$$dT^2 = \omega^2 dn^2 \quad (288)$$

and hence (286) becomes

$$Y = \int_{T'} \int_{T''} F(T) dT^2 = \omega^2 \int_{n'} \int_{n''} F(t + n\omega) dn^2 \quad (289)$$

upon which relation the subsequent formulae are based.

76. *Double Integration as Based upon NEWTON'S Formula of Interpolation.*—If we substitute, successively,  $n'$  and  $n''$  for  $n$  in (243), and take the difference of the two results, we obtain

$$\int_{n'}^{n''} F(t + n\omega) dn = \Psi(n'') - \Psi(n') \quad (290)$$

From the *form* of (290) it follows that the expression for the *indefinite* integral is —

$$\int F(t+n\omega)dn = \Psi(n)$$

or, by (238),

$$\int F(t+n\omega)dn = \int F_n dn = {}^{\prime}F_n + \tfrac{1}{2} F_n + \beta \Delta^{\prime}_n + \gamma \Delta^{\prime\prime}_n + \delta \Delta^{\prime\prime\prime}_n + \dots \tag{291}$$

the constant of integration being contained in  ${}^{\prime}F_n$ , which depends upon the arbitrary quantity  ${}^{\prime}F_0$ . Multiplying this equation by  $dn$ , and integrating, we get

$$\int \int F(t+n\omega)dn^2 = \int {}^{\prime}F_n dn + \tfrac{1}{2} \int F_n dn + \beta \int \Delta^{\prime}_n dn + \gamma \int \Delta^{\prime\prime}_n dn + \delta \int \Delta^{\prime\prime\prime}_n dn + \dots \tag{292}$$

Let us now consider a new series, namely —

$${}^{\prime\prime}F_0, {}^{\prime\prime}F_1, {}^{\prime\prime}F_2, {}^{\prime\prime}F_3, \dots {}^{\prime\prime}F_{i+2}$$

the term  ${}^{\prime\prime}F_0$  being arbitrary, and the subsequent terms so determined that the quantities

$${}^{\prime}F_0, {}^{\prime}F_1, {}^{\prime}F_2, \dots {}^{\prime}F_{i+1}$$

are the successive first differences of the proposed series. The manner of arranging the series  ${}^{\prime\prime}F$ ,  ${}^{\prime}F$ , and  $F$ , together with the differences of  $F$ , is shown in the schedule below :

$T$	${}^{\prime\prime}F$	${}^{\prime}F$	$F(T)$	$\Delta^{\prime}$	$\Delta^{\prime\prime}$	$\Delta^{\prime\prime\prime}$	$\Delta^{\text{iv}}$
$t$	${}^{\prime\prime}F_0$						
$t + \omega$	${}^{\prime\prime}F_1$	${}^{\prime}F_0$	$F_0$	$\Delta^{\prime}_0$			
$t + 2\omega$	${}^{\prime\prime}F_2$	${}^{\prime}F_1$	$F_1$	$\Delta^{\prime}_1$	$\Delta^{\prime\prime}_0$		
$t + 3\omega$	${}^{\prime\prime}F_3$	${}^{\prime}F_2$	$F_2$	$\Delta^{\prime}_2$	$\Delta^{\prime\prime}_1$	$\Delta^{\prime\prime\prime}_0$	$\Delta^{\text{iv}}_0$
	${}^{\prime\prime}F_4$	${}^{\prime}F_3$	$F_3$		$\Delta^{\prime\prime}_2$	$\Delta^{\prime\prime\prime}_1$	$\Delta^{\text{iv}}_1$
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$t + (i-2)\omega$	${}^{\prime\prime}F_{i-1}$	.	.	.	.	.	.
$t + (i-1)\omega$	${}^{\prime\prime}F_i$	${}^{\prime}F_{i-1}$	$F_{i-2}$	$\Delta^{\prime}_{i-2}$	$\Delta^{\prime\prime}_{i-3}$	.	$\Delta^{\text{iv}}_{i-4}$
$t + i\omega$	${}^{\prime\prime}F_{i+1}$	${}^{\prime}F_i$	$F_{i-1}$	$\Delta^{\prime}_{i-1}$	$\Delta^{\prime\prime}_{i-2}$	$\Delta^{\prime\prime\prime}_{i-3}$	
	${}^{\prime\prime}F_{i+2}$	${}^{\prime}F_{i+1}$	$F_i$				

Now, since the differences  $\Delta^{(r)}$  may be regarded as a series of functions whose 1st, 2d, . . . . differences are  $\Delta^{(r+1)}$ ,  $\Delta^{(r+2)}$  . . . . , it is clear that formula (291) may be applied successively to each of the integrals in the second member of (292). Accordingly, we have

$$\left. \begin{aligned} \int {}^1F_n dn &= {}''F_n + \tfrac{1}{2} {}^1F_n + \beta F_n + \gamma \Delta'_n + \delta \Delta''_n + \epsilon \Delta'''_n + \dots \\ \tfrac{1}{2} \int F_n dn &= \tfrac{1}{2} ({}^1F_n + \tfrac{1}{2} F_n + \beta \Delta'_n + \gamma \Delta''_n + \delta \Delta'''_n + \dots) \\ \beta \int \Delta'_n dn &= \beta (F_n + \tfrac{1}{2} \Delta'_n + \beta \Delta''_n + \gamma \Delta'''_n + \dots) \\ \gamma \int \Delta''_n dn &= \gamma (\Delta'_n + \tfrac{1}{2} \Delta''_n + \beta \Delta'''_n + \dots) \\ \delta \int \Delta'''_n dn &= \delta (\Delta''_n + \tfrac{1}{2} \Delta'''_n + \dots) \\ \epsilon \int \Delta^{iv}_n dn &= \epsilon (\Delta'''_n + \dots) \\ \dots &\dots \end{aligned} \right\} \quad (293)$$

Summing these expressions, we find, in accordance with (292),

$$\iint F(t+n\omega) dn^2 = {}''F_n + {}^1F_n + (\tfrac{1}{4} + 2\beta) F_n + (\beta + 2\gamma) \Delta'_n + (\beta^2 + \gamma + 2\delta) \Delta''_n + (2\beta\gamma + \delta + 2\epsilon) \Delta'''_n + \dots \quad (294)$$

Upon substituting the numerical values of  $\beta$ ,  $\gamma$ ,  $\delta$ , . . . . from (222), formula (294) becomes

$$\iint F(t+n\omega) dn^2 = {}''F_n + {}^1F_n + \tfrac{1}{12} F_n - \tfrac{1}{240} \Delta''_n + \tfrac{1}{240} \Delta'''_n - \dots \quad (294a)$$

the coefficient of  $\Delta'_n$  reducing to zero. We proceed to determine the expansion to which the coefficients of this formula belong. For brevity, let us write (294) in the form

$$\iint F(t+n\omega) dn^2 = {}''F_n + {}^1F_n + aF_n + b\Delta'_n + c\Delta''_n + d\Delta'''_n + \dots \quad (295)$$

Now, from (228), we have

$$\frac{1}{\log(1+x)} = x^{-1} + \tfrac{1}{2} x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots \quad (296)$$

Also, let us put

$$w \equiv x^{-2} + x^{-1} + ax^0 + bx + cx^2 + dx^3 + \dots \quad (297)$$

in which the coefficients are taken as in (295). Whence, since the second member of (295) is the combined sum of the second members in (293), it is evident that (297) may be resolved, conversely, as follows :

$$\begin{aligned} w = & x^{-2} + \frac{1}{2}x^{-1} + \beta x^0 + \gamma x + \delta x^2 + \dots \\ & + \frac{1}{2}(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \dots) \\ & + \beta(x^0 + \frac{1}{2}x + \beta x^2 + \dots) \\ & + \gamma(x + \frac{1}{2}x^2 + \dots) \\ & + \delta(x^2 + \dots) \\ & + \dots \end{aligned}$$

which may be written

$$\begin{aligned} w = & x^{-1}(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots) \\ & + \frac{1}{2}x^0(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots) \\ & + \beta x(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots) \\ & + \gamma x^2(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots) \\ & + \delta x^3(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots) \\ & + \dots \\ = & (x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \dots)(x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \dots) \\ = & (x^{-1} + \frac{1}{2}x^0 + \beta x + \gamma x^2 + \delta x^3 + \dots)^2 \end{aligned}$$

Therefore, by (296), we have

$$\begin{aligned} w = & \left\{ \log(1+x) \right\}^{-2} = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right)^{-2} \\ = & x^{-2} + x^{-1} + \frac{1}{1 \cdot 2}x^0 - \frac{1}{2 \cdot 4 \cdot 6}x^2 + \frac{1}{2 \cdot 4 \cdot 6}x^3 - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6}x^4 + \frac{1 \cdot 9}{6 \cdot 6 \cdot 4 \cdot 8}x^5 - \dots \end{aligned} \quad (298)$$

Comparing (297) and (298), it follows that the coefficients of the former, and hence, also, those of (295), are the coefficients in the expansion of  $[\log(1+x)]^{-2}$ , as developed in (298). Whence, introducing these values of  $a, b, c, d, \dots$  in (295), we obtain

$$\iint F(t+n\omega) dn^2 = {}''F_n + {}'F_n + \frac{1}{1 \cdot 2}F_n - \frac{1}{2 \cdot 4 \cdot 6}A_n'' + \frac{1}{2 \cdot 4 \cdot 6}A_n''' - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6}A_n^{iv} + \frac{1 \cdot 9}{6 \cdot 6 \cdot 4 \cdot 8}A_n^v - \dots \quad (299)$$

as was found directly—in part—in (294a).

Let us now put

$$\begin{aligned} \lambda(n) = & {}''F_n + {}'F_n + aF_n + bA_n' + cA_n'' + dA_n''' + eA_n^{iv} + \dots \\ = & {}''F_n + {}'F_n + \frac{1}{1 \cdot 2}F_n + 0A_n' - \frac{1}{2 \cdot 4 \cdot 6}A_n'' + \frac{1}{2 \cdot 4 \cdot 6}A_n''' - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6}A_n^{iv} + \dots \end{aligned} \quad (300)$$

and (299) becomes

$$\iint F(t+n\omega) dn^2 = \lambda(n) \quad (301)$$

Whence, if the integral be taken between the two fractional limits,  $n'$  and  $n''$ , we shall have

$$\int \int_{n'}^{n''} F(t+n\omega) dn^2 = \lambda(n'') - \lambda(n') \quad (302)$$

And if we make the upper limit an integer, say  $n'' = i$ , we have

$$\int \int_{n'}^i F(t+n\omega) dn^2 = \lambda(i) - \lambda(n') \quad (303)$$

The last formula involves the disadvantage of employing differences  $\Delta_i', \Delta_i'', \Delta_i''', \dots$  which are not given when the tabulation of  $F(T)$  ends with the quantity  $F_i$ . To remedy this defect, we proceed as follows: Put

$$v = \lambda(i) = {}''F_i + {}'F_i + aF_i + b\Delta_i' + c\Delta_i'' + d\Delta_i''' + e\Delta_i^{iv} + \dots \quad (304)$$

and substitute for  ${}''F_i, {}'F_i, F_i, \Delta_i', \Delta_i'', \dots$  the expressions

$$\left. \begin{aligned} {}''F_i &= {}''F_{i+2} - 2{}'F_{i+1} + F_i \\ {}'F_i &= {}'F_{i+1} - F_i \\ F_i &= F_i \\ \Delta_i' &= \Delta_{i-1}' + \Delta_{i-2}' + \Delta_{i-3}'' + \Delta_{i-4}^{iv} + \dots \\ \Delta_i'' &= \Delta_{i-2}' + 2\Delta_{i-3}'' + 3\Delta_{i-4}^{iv} + \dots \\ \Delta_i''' &= \Delta_{i-3}'' + 3\Delta_{i-4}^{iv} + \dots \\ \Delta_i^{iv} &= \Delta_{i-4}^{iv} + \dots \\ &\dots \end{aligned} \right\} \quad (305)$$

Whence the integral (303) may at once be expressed in terms of the *available* differences,  $\Delta_{i-1}', \Delta_{i-2}'', \Delta_{i-3}''', \dots$ . However, to avoid direct substitution, let us put, as in (229),

$$x = \frac{u}{1-u} \quad (306)$$

and we shall have

$$\left. \begin{aligned} x^{-2} &= u^{-2}(1-u)^2 = u^{-2} - 2u^{-1} + u^0 \\ x^{-1} &= u^{-1}(1-u) = u^{-1} - u^0 \\ x^0 &= u^0 \\ x &= u(1-u)^{-1} = u + u^2 + u^3 + u^4 + \dots \\ x^2 &= u^2(1-u)^{-2} = u^2 + 2u^3 + 3u^4 + \dots \\ x^3 &= u^3(1-u)^{-3} = u^3 + 3u^4 + \dots \\ x^4 &= u^4(1-u)^{-4} = u^4 + \dots \\ &\dots \end{aligned} \right\} \quad (307)$$

Again, from (297), we have

$$w = x^{-2} + x^{-1} + ax^0 + bx + cx^2 + dx^3 + ex^4 + \dots \quad (308)$$

Now, it is evident that if the expressions (307) be substituted in the second member of (308), the algebraic process will be identical in form with that of substituting the expressions (305) in (304). The  $w$  operation involves the quantities

$$w ; x^{-2}, x^{-1}, x^0, x, x^2, x^3, \dots ; u^{-2}, u^{-1}, u^0, u, u^2, u^3, \dots ;$$

while the  $v$  operation involves, in exactly the same manner, the quantities

$$v ; {}''F_i, {}'F_i, F_i, A'_i, A''_i, A'''_i, \dots ; {}''F_{i+2}, {}'F_{i+1}, F_i, A'_{i-1}, A''_{i-2}, A'''_{i-3}, \dots ;$$

Hence, if we perform the  $w$  operation, the result for  $v$  is at once known. But the expression which results from substituting (307) in (308) is obtained with greater expedition by the following process : From (298), we have

$$w = \{\log(1+x)\}^{-2}$$

Whence, by (306), we find

$$w = \{-\log(1-u)\}^{-2} = \{\log(1-u)\}^{-2}$$

the expansion of which is immediately obtained by writing  $-u$  for  $x$  in the second member of (297). Thus we find

$$w = u^{-2} - u^{-1} + au^0 - bu + cu^2 - du^3 + eu^4 - \dots \quad (309)$$

Therefore, according to the preceding reasoning, the expression for  $v$  is —

$$v = {}''F_{i+2} - {}'F_{i+1} + aF_i - bA'_{i-1} + cA''_{i-2} - dA'''_{i-3} + eA^{iv}_{i-4} - \dots$$

Denoting this expression by  $\pi(i)$ , and restoring the numerical values of  $a, b, c, \dots$  from (300), we have

$$\begin{aligned} v = \pi(i) &= {}''F_{i+2} - {}'F_{i+1} + aF_i - bA'_{i-1} + cA''_{i-2} - dA'''_{i-3} + eA^{iv}_{i-4} - \dots \\ &= {}''F_{i+2} - {}'F_{i+1} + \frac{1}{1 \cdot 2} F_i - \frac{1}{2 \cdot 4 \cdot 6} A''_{i-2} - \frac{1}{2 \cdot 4 \cdot 6} A'''_{i-3} - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6} A^{iv}_{i-4} - \dots \end{aligned} \quad (310)$$

Whence, by (304) and (310),

$$\lambda(i) = v = \pi(i)$$

and the formula (303) becomes, therefore,

$$\int \int_{n'}^i F(t+n\omega) dn^2 = \pi(i) - \lambda(n') \quad (311)$$

In the formula just proved the quantity  $i$  denotes an integer. Now, by the general method of interpolation employed in §70, it is easily shown that (311) is true for non-integral values of  $i$ . Thus, writing  $n''$  for  $i$ , this formula becomes

$$\int \int_{n'}^{n''} F(t+n\omega) dn^2 = \pi(n'') - \lambda(n') \quad (312)$$

We now bring together equations (300), (310), (302), (312) and (289), in the order named; observing that in the first two of these we may write  $''F_{n+1}$  for  $''F_n + 'F_n$  and for  $''F_{n+2} - 'F_{n+1}$ , respectively. Thus we obtain the following group :

$$\left. \begin{aligned} \lambda(n) &= ''F_{n+1} + \frac{1}{12} F_n - \frac{1}{240} A_n'' + \frac{1}{240} A_n''' - \frac{221}{60480} A_n^{iv} + \frac{19}{6048} A_n^v - \dots \\ \pi(n) &= ''F_{n+1} + \frac{1}{12} F_n - \frac{1}{240} A_n'' - \frac{1}{240} A_n''' - \frac{221}{60480} A_n^{iv} - \frac{19}{6048} A_n^v - \dots \\ &\quad \int \int_{n'}^{n''} F(t+n\omega) dn^2 = \lambda(n'') - \lambda(n') \\ &\quad \int \int_{n'}^{n''} F(t+n\omega) dn^2 = \pi(n'') - \lambda(n') \\ Y &= \int \int_{t+n'\omega}^{t+n''\omega} F(T) dT^2 = \omega^2 \int \int_{n'}^{n''} F(t+n\omega) dn^2 \end{aligned} \right\} \quad (313)$$

From this group are immediately derived all of the formulae given in the following section.

77. We have already remarked that in the process of single integration the value of the definite integral is wholly independent of the absolute value of  $'F_0$ , which may therefore be assigned arbitrarily. Similarly, in double integration, the quantity  $''F_0$  may be taken at pleasure, the integral being independent of its absolute value. Per contra, the *double* integral will evidently vary with the value assigned to  $'F_0$ . Hence, unless  $'F_0$  is fixed by some special consideration, the value of the double integral is indeterminate—a conclusion already derived from (287).

Now, as was previously remarked, the value of the first integral corresponding to the lower limit is usually known in practical applications. We shall therefore denote by  $H_0$  the value of  $\int F(T) dT$  which results when  $t$  is substituted for  $T$ . Then, by (291), we have

$$\begin{aligned} H_0 &= \left[ \int F(T) dT \right]_{T=t} = \omega \left[ \int F(t+n\omega) dn \right]_{n=0} \\ &= \omega (F_0 + \frac{1}{2} F_0' + \beta A_0'' + \gamma A_0''' + \delta A_0^{iv} + \epsilon A_0^v + \dots) \\ &= \omega (F_1 - \frac{1}{2} F_0' + \beta A_0'' + \gamma A_0''' + \delta A_0^{iv} + \epsilon A_0^v + \dots) \end{aligned}$$

or, upon restoring the numerical values of  $\beta, \gamma, \delta, \dots$  from (222), and transposing,

$$'F_1 = \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A_0' - \frac{1}{24} A_0'' + \frac{1}{720} A_0''' - \frac{1}{160} A_0^{iv} + \frac{8}{60480} A_0^v - \dots \quad (314)$$

which determines  $'F_1$ , and hence, also, the double integral  $Y$ , provided  $H_0$  is known. In practice the value of  $H_0$  is frequently zero.

Using (314) in conjunction with the relations (313), we obtain the several groups of quadrature formulae given below :

$$\left. \begin{aligned} 'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A_0' - \frac{1}{24} A_0'' + \frac{1}{720} A_0''' - \frac{1}{160} A_0^{iv} + \frac{8}{60480} A_0^v - \dots \\ \int \int_t^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^1 \int_0^1 F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_{i+1} - {}''F_1) + \frac{1}{12} (F_i - F_0) - \frac{1}{240} (A_i'' - A_0'') + \frac{1}{240} (A_i''' - A_0''') \\ &\quad - \frac{2}{60480} (A_i^{iv} - A_0^{iv}) + \frac{1}{6048} (A_i^v - A_0^v) - \dots \} \end{aligned} \right\} \quad (315)$$

$$\left. \begin{aligned} 'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A_0' - \frac{1}{24} A_0'' + \frac{1}{720} A_0''' - \frac{1}{160} A_0^{iv} + \frac{8}{60480} A_0^v - \dots \\ \int \int_t^{t+n\omega} F(T) dT^2 &= \omega^2 \int_0^n \int_0^n F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_{n+1} - {}''F_1) + \frac{1}{12} (F_n - F_0) - \frac{1}{240} (A_n'' - A_0'') + \frac{1}{240} (A_n''' - A_0''') \\ &\quad - \frac{2}{60480} (A_n^{iv} - A_0^{iv}) + \frac{1}{6048} (A_n^v - A_0^v) - \dots \} \end{aligned} \right\} \quad (316)$$

$$\left. \begin{aligned} 'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A_0' - \frac{1}{24} A_0'' + \frac{1}{720} A_0''' - \frac{1}{160} A_0^{iv} + \frac{8}{60480} A_0^v - \dots \\ \int \int_{t+n\omega}^{t+\omega} F(T) dT^2 &= \omega^2 \int_n^1 \int_n^1 F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_{i+1} - {}''F_{n+1}) + \frac{1}{12} (F_i - F_n) - \frac{1}{240} (A_i'' - A_n'') + \frac{1}{240} (A_i''' - A_n''') \\ &\quad - \frac{2}{60480} (A_i^{iv} - A_n^{iv}) + \frac{1}{6048} (A_i^v - A_n^v) - \dots \} \end{aligned} \right\} \quad (317)$$

$$\left. \begin{aligned} 'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A_0' - \frac{1}{24} A_0'' + \frac{1}{720} A_0''' - \frac{1}{160} A_0^{iv} + \frac{8}{60480} A_0^v - \dots \\ \int \int_{t+n'\omega}^{t+n''\omega} F(T) dT^2 &= \omega^2 \int_{n'}^{n''} \int_{n'}^{n''} F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_{n''+1} - {}''F_{n'+1}) + \frac{1}{12} (F_{n''} - F_{n'}) - \frac{1}{240} (A_{n''}'' - A_{n'}'') + \frac{1}{240} (A_{n''}''' - A_{n'}''') \\ &\quad - \frac{2}{60480} (A_{n''}^{iv} - A_{n'}^{iv}) + \frac{1}{6048} (A_{n''}^v - A_{n'}^v) - \dots \} \end{aligned} \right\} \quad (318)$$

The foregoing formulae are applicable when the upper limit falls near the *beginning* of the tabular series. When the upper limits falls at or near the *end* of the given series, the following formulae—likewise derived from (313)—may be employed :

$$\left. \begin{aligned} {}^iF_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{1}{720} A'''_0 - \frac{1}{160} A^{iv}_0 + \frac{8}{60480} A^v_0 - \dots \\ \int \int_t^{t+i\omega} F(T) dT^2 &= \omega^2 \int_0^i F(t+n\omega) dn^2 \\ &= \omega^2 \left\{ ({}^iF_{i+1} - {}^iF_1) + \frac{1}{12} (F_i - F_0) - \frac{1}{240} (A''_{i-2} - A''_0) - \frac{1}{240} (A'''_{i-3} + A'''_0) \right. \\ &\quad \left. - \frac{2}{60480} (A^{iv}_{i-4} - A^{iv}_0) - \frac{1}{6048} (A^v_{i-5} + A^v_0) - \dots \right\} \end{aligned} \right\} \quad (319)$$

$$\left. \begin{aligned} {}^iF_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{1}{720} A'''_0 - \frac{1}{160} A^{iv}_0 + \frac{8}{60480} A^v_0 - \dots \\ \int \int_t^{t+i\omega} F(T) dT^2 &= \omega^2 \int_0^n F(t+n\omega) dn^2 \\ &= \omega^2 \left\{ ({}^iF_{n+1} - {}^iF_1) + \frac{1}{12} (F_n - F_0) - \frac{1}{240} (A''_{n-2} - A''_0) - \frac{1}{240} (A'''_{n-3} + A'''_0) \right. \\ &\quad \left. - \frac{2}{60480} (A^{iv}_{n-4} - A^{iv}_0) - \frac{1}{6048} (A^v_{n-5} + A^v_0) - \dots \right\} \end{aligned} \right\} \quad (320)$$

$$\left. \begin{aligned} {}^iF_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{1}{720} A'''_0 - \frac{1}{160} A^{iv}_0 + \frac{8}{60480} A^v_0 - \dots \\ \int \int_{t+n\omega}^{t+i\omega} F(T) dT^2 &= \omega^2 \int_n^i F(t+n\omega) dn^2 \\ &= \omega^2 \left\{ ({}^iF_{i+1} - {}^iF_{n+1}) + \frac{1}{12} (F_i - F_n) - \frac{1}{240} (A''_{i-2} - A''_n) - \frac{1}{240} (A'''_{i-3} + A'''_n) \right. \\ &\quad \left. - \frac{2}{60480} (A^{iv}_{i-4} - A^{iv}_n) - \frac{1}{6048} (A^v_{i-5} + A^v_n) - \dots \right\} \end{aligned} \right\} \quad (321)$$

$$\left. \begin{aligned} {}^iF_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{1}{720} A'''_0 - \frac{1}{160} A^{iv}_0 + \frac{8}{60480} A^v_0 - \dots \\ \int \int_{t+n'\omega}^{t+n''\omega} F(T) dT^2 &= \omega^2 \int_{n'}^{n''} F(t+n\omega) dn^2 \\ &= \omega^2 \left\{ ({}^iF_{n''+1} - {}^iF_{n'+1}) + \frac{1}{12} (F_{n''} - F_{n'}) - \frac{1}{240} (A''_{n''-2} - A''_{n'}) - \frac{1}{240} (A'''_{n''-3} + A'''_{n'}) \right. \\ &\quad \left. - \frac{2}{60480} (A^{iv}_{n''-4} - A^{iv}_{n'}) - \frac{1}{6048} (A^v_{n''-5} + A^v_{n'}) - \dots \right\} \end{aligned} \right\} \quad (322)$$

In applications of all the preceding formulae, the value of  ${}^iF_1$  (or of  ${}^iF_0$  when employed) is wholly arbitrary, and therefore may be assigned at pleasure in every case. But when (315), (316), (319) and (320) are applicable, it is frequently convenient to determine  ${}^iF_1$  such that

$$-{}^iF_1 - \frac{1}{12} F_0 + \frac{1}{240} A''_0 - \frac{1}{240} A'''_0 + \frac{2}{60480} A^{iv}_0 - \frac{1}{6048} A^v_0 + \dots = 0$$

The formulae in question then take the form as follows :

$$\left. \begin{aligned}
 {}'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{19}{720} A'''_0 - \frac{3}{160} A^{iv}_0 + \frac{863}{60480} A^v_0 - \dots \\
 {}''F_1 &= -\frac{1}{12} F_0 + \frac{1}{240} A'_0 - \frac{1}{240} A''_0 + \frac{221}{60480} A^{iv}_0 - \frac{19}{6048} A^v_0 + \dots \\
 \int \int_t^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^i F(t+n\omega) dn^2 \\
 &= \omega^2 ({}''F_{i+1} + \frac{1}{12} F_i - \frac{1}{240} A'_i + \frac{1}{240} A''_i - \frac{221}{60480} A^{iv}_i + \frac{19}{6048} A^v_i - \dots)
 \end{aligned} \right\} \quad (323)$$

$$\left. \begin{aligned}
 {}'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{19}{720} A'''_0 - \frac{3}{160} A^{iv}_0 + \frac{863}{60480} A^v_0 - \dots \\
 {}''F_1 &= -\frac{1}{12} F_0 + \frac{1}{240} A'_0 - \frac{1}{240} A''_0 + \frac{221}{60480} A^{iv}_0 - \frac{19}{6048} A^v_0 + \dots \\
 \int \int_t^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^n F(t+n\omega) dn^2 \\
 &= \omega^2 ({}''F_{n+1} + \frac{1}{12} F_n - \frac{1}{240} A'_n + \frac{1}{240} A''_n - \frac{221}{60480} A^{iv}_n + \frac{19}{6048} A^v_n - \dots)
 \end{aligned} \right\} \quad (324)$$

$$\left. \begin{aligned}
 {}'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{19}{720} A'''_0 - \frac{3}{160} A^{iv}_0 + \frac{863}{60480} A^v_0 - \dots \\
 {}''F_1 &= -\frac{1}{12} F_0 + \frac{1}{240} A'_0 - \frac{1}{240} A''_0 + \frac{221}{60480} A^{iv}_0 - \frac{19}{6048} A^v_0 + \dots \\
 \int \int_t^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^i F(t+n\omega) dn^2 \\
 &= \omega^2 ({}''F_{i+1} + \frac{1}{12} F_i - \frac{1}{240} A'_{i-2} - \frac{1}{240} A''_{i-3} - \frac{221}{60480} A^{iv}_{i-4} - \frac{19}{6048} A^v_{i-5} - \dots)
 \end{aligned} \right\} \quad (325)$$

$$\left. \begin{aligned}
 {}'F_1 &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} A'_0 - \frac{1}{24} A''_0 + \frac{19}{720} A'''_0 - \frac{3}{160} A^{iv}_0 + \frac{863}{60480} A^v_0 - \dots \\
 {}''F_1 &= -\frac{1}{12} F_0 + \frac{1}{240} A'_0 - \frac{1}{240} A''_0 + \frac{221}{60480} A^{iv}_0 - \frac{19}{6048} A^v_0 + \dots \\
 \int \int_t^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^n F(t+n\omega) dn^2 \\
 &= \omega^2 ({}''F_{n+1} + \frac{1}{12} F_n - \frac{1}{240} A'_{n-2} - \frac{1}{240} A''_{n-3} - \frac{221}{60480} A^{iv}_{n-4} - \frac{19}{6048} A^v_{n-5} - \dots)
 \end{aligned} \right\} \quad (326)$$

The differences which appear in the foregoing formulae, together with the auxiliary functions  $'F$  and  $''F$ , are to be taken according to the schedule on page 161. The symbol  $i$  denotes a positive integer, while  $n$  designates a fractional or mixed number: so that all functions and differences whose subscripts involve  $n$  must be derived from their respective series by *interpolation*. Finally, the quantity  $H_0$  denotes—as previously defined—the value of  $\int F(T) dT$  when  $t$  is substituted for  $T$ : so that we have

$$H_0 = \left[ \int F(T) dT \right]_{T=t} \quad (327)$$

It may happen occasionally that the value of  $H_0$  is unknown, while the value of  $\int F(T) dT$  corresponding to  $T = t + n\omega$  is known for a particular value of  $n$ . Denoting this quantity by  $H_n$ ,

we may, by any one of the foregoing methods, compute the definite integral

$$X = \int_t^{t+n\omega} F(T) dT = H_n - H_0$$

and hence find

$$H_0 = H_n - X$$

(327a)

with which value we proceed as before.

Several examples will now be solved as an exercise to illustrate the formulae given above.

EXAMPLE I.—Let it be required to find

$$Y = \int_0^{\frac{\pi}{2}} \int_{\cos T}^{\cos T} T dT^2$$

on the supposition that  $\int \cos T dT = 2$  when  $T = 0$ .

We tabulate and difference the following values of  $F(T) \equiv \cos T$ :

$T$	$''F$	$'F$	$F(T) \equiv \cos T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
0	0.00000	11.95916	1.00000	— 1519	— 2993	—	—
10	11.95916	12.94397	0.98481	4512	2854	+ 139	—
20	24.90313	13.88366	0.93969	7366	2633	221	+ 82
30	38.78679	14.74969	0.86603	9999	2326	307	86
40	53.53648	15.51573	0.76604	12325	1954	372	65
50	69.05221	16.15852	0.64279	14279	1519	435	63
60	85.21073	16.65852	0.50000	15798	1039	480	45
70	101.86925	17.00054	0.34202	16837	— 528	+ 511	+ 31
80	118.86979	17.17419	0.17365	— 17365	—	—	—
90	136.04398	—	0.00000	—	—	—	—

Accordingly, we have

$$t = 0^\circ \qquad \omega = 10^\circ = \frac{\pi}{18} \qquad H_0 = 2 \qquad i = 9$$

Proceeding by (319), the computation of  $'F_1$  is as follows:

$$H_0 \div \omega = +11.45915.6$$

$$F_0 = +1.00000 \qquad + \frac{1}{2} F_0 = + 0.50000.0$$

$$\Delta'_0 = - 1519 \qquad + \frac{1}{12} \Delta'_0 = - 126.6$$

$$\Delta''_0 = - 2993 \qquad - \frac{1}{24} \Delta''_0 = + 124.7$$

$$\Delta'''_0 = + 139 \qquad + \frac{19}{720} \Delta'''_0 = + 3.7$$

$$\Delta^{iv}_0 = + 82 \qquad - \frac{3}{160} \Delta^{iv}_0 = - 1.5$$

$$\therefore 'F_1 = +11.95916$$

The column  $'F$  is now completed by successive additions ; hence, also, the column  $''F$ , having first assumed  $''F_1 = 0$ . Whence, by (319), the remainder of the computation is as follows :

$''F_{10} = +136.04398$	$''F_1 = 0.00000$	$(''F_{10} - ''F_1) = +136.04398$
$F_9 = 0.00000$	$F_0 = +1.00000$	$+ \frac{1}{12} (F_9 - F_0) = - 0.08333.3$
$\Delta_7'' = - 528$	$\Delta_0'' = - 2993$	$- \frac{1}{240} (\Delta_7'' - \Delta_0'') = - 10.3$
$\Delta_6''' = + 511$	$\Delta_0''' = + 139$	$- \frac{1}{240} (\Delta_6''' + \Delta_0''') = - 2.7$
$\Delta_5^{iv} = + 31$	$\Delta_0^{iv} = + 82$	$- \frac{2}{60} \frac{2}{4} \frac{1}{80} (\Delta_5^{iv} - \Delta_0^{iv}) = + 0.2$
$\log \Sigma = 2.1334129$		$\Sigma = +135.96052$
$\log \omega^2 = 8.4837548$		
$\log Y = 0.6171677$		$\therefore Y = 4.141595$

To verify this result, we have

$$\int \cos TdT = \sin T + C$$
$$Y = \int \int_0^{\frac{\pi}{2}} \cos TdT^2 = \left[ -\cos T + CT \right]_0^{\frac{\pi}{2}} = 1 + \frac{1}{2} C\pi$$

where  $C$  is the constant of the first integration. To determine  $C$ , the first of these relations gives

$$H_0 = \left[ \sin T + C \right]_{T=0} = C$$

whence

$$C = 2$$

and therefore

$$Y = 1 + \pi = 4.141593$$

EXAMPLE II.—Compute the value of

$$Y = \int \int_2^{2.463} T^{-2} dT^2$$

which corresponds to  $H_0 = 0$ .

Here we tabulate and difference  $F(T) \equiv T^{-2}$  as below :

$T$	$''F$	$'F$	$F(T) \equiv T^{-2}$	$\Delta'$	$\Delta''$	$\Delta'''$
2.0	-0.02082		0.25000			
2.1	+0.10210	+0.12292	0.22676	-2324	+309	
2.2	0.45178	0.34968	0.20661	2015	258	-51
2.3	1.00807	0.55629	0.18904	1757	214	44
2.4	1.75340	0.74533	0.17361	1543	+182	-32
2.5	+2.67234	+0.91894	0.16000	-1361		

We have, therefore,

$$t = 2.0 \qquad \omega = 0.1 \qquad H_0 = 0$$

whence, proceeding by (326), the computation of  $'F_1$  and  $''F_1$  is as follows :

$+ \frac{1}{2} F_0 = +0.12500$	$\dots\dots\dots$	$- \frac{1}{12} F_0 = -0.02083.3$
$+ \frac{1}{12} A'_0 = -193.7$		$+ \frac{1}{240} A''_0 = +1.3$
$- \frac{1}{24} A''_0 = -12.9$		$- \frac{1}{240} A'''_0 = +0.2$
$+ \frac{1.9}{720} A'''_0 = -1.3$		
$\therefore 'F_1 = +0.12292$		$\therefore ''F_1 = -0.02082$

From the completed table we now find

$n = (2.468 - 2.0) \div 0.1$	$\dots\dots\dots$	
$= 4.68 = 5 - 0.32$		$''F_{n+1} = +2.36025.6$
$F_n = +0.16418$	$+ \frac{1}{12} F_n = +1368.2$	
$A''_{n-2} = +191$	$- \frac{1}{240} A''_{n-2} = -0.8$	
$A'''_{n-3} = -36$	$- \frac{1}{240} A'''_{n-3} = +0.1$	
	$\Sigma = +2.37393$	
	$\therefore Y = +0.0237393$	

This result is easily verified, for we have

$$\int T^{-2}dT = -\frac{1}{T} + C$$

$$Y = \left[ -\log_e T + CT \right]_2^{2.468} = -\log_e 1.234 + 0.468C$$

also

$$0 = H_0 = \left[ -\frac{1}{T} + C \right]_{T=2} = -\frac{1}{2} + C$$
$$\therefore C = \frac{1}{2}$$

Hence

$$Y = -\log_e 1.234 + 0.234 = -0.2102609 + 0.234 = +0.0237391$$

with which the above result substantially agrees.

EXAMPLE III.—From the table of the preceding example, find the value of

$$Y = \int_2^{2.15} \int_2^{2.15} T^{-2}dT^2$$

Here we employ formula (324), in which we take

$$n = \frac{2.15 - 2.0}{0.1} = 1.50 = 1 + \frac{1}{2}$$

We therefore obtain

$(n + 1 = 2 + \frac{1}{2})$	${}''F_{n+1} = +0.24992.0$
$F_n = +0.21633$	$+ \frac{1}{12} F_n = + 1802.8$
$\Delta'_n = + 235$	$- \frac{1}{240} \Delta''_n = - 1.0$
$\Delta'''_n = - 38$	$+ \frac{1}{240} \Delta'''_n = - 0.2$
	$\Sigma = +0.26794$
	$\therefore Y = +0.0026794$

The true mathematical value of  $Y$  is—

$$Y = 0.075 - \log_e 1.075 = +0.0026793 \dots$$

78. *Double Integration as Based upon STIRLING'S and BESSEL'S Formulae of Interpolation.*—Let the schedule of functions (including  $'F$  and  $''F$ ) and differences to be used in the subsequent formulae of quadrature be as follows :

$T$	${}''F$	$'F$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$
$t - 2\omega$			$F_{-2}$		$\Delta''_{-2}$	$\Delta'''_{-\frac{3}{2}}$
$t - \omega$	${}''F_{-1}$	$'F_{-\frac{1}{2}}$	$F_{-1}$	$\Delta'_{-\frac{1}{2}}$	$\Delta''_{-1}$	$\Delta'''_{-\frac{1}{2}}$
$t$	${}''F_0$	$'F_{\frac{1}{2}}$	$F_0$	$\Delta'_{\frac{1}{2}}$	$\Delta''_0$	$\Delta'''_{\frac{1}{2}}$
$t + \omega$	${}''F_1$	$'F_{\frac{3}{2}}$	$F_1$	$\Delta'_{\frac{3}{2}}$	$\Delta''_1$	$\Delta'''_{\frac{3}{2}}$
$t + 2\omega$	${}''F_2$	.	$F_2$	.	$\Delta''_2$	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$t + (i-1)\omega$	${}''F_{i-1}$	$'F_{i-\frac{1}{2}}$	$F_{i-1}$	$\Delta'_{i-\frac{1}{2}}$	$\Delta''_{i-1}$	$\Delta'''_{i-\frac{1}{2}}$
$t + i\omega$	${}''F_i$	$'F_{i+\frac{1}{2}}$	$F_i$	$\Delta'_{i+\frac{1}{2}}$	$\Delta''_i$	$\Delta'''_{i+\frac{1}{2}}$
$t + (i+1)\omega$	${}''F_{i+1}$		$F_{i+1}$	$\Delta'_{i+\frac{3}{2}}$	$\Delta''_{i+1}$	$\Delta'''_{i+\frac{3}{2}}$
$t + (i+2)\omega$			$F_{i+2}$		$\Delta''_{i+2}$	

From the form of (263) it follows that the expression for the indefinite integral of  $F(t + n\omega) \, dn$  is—

$$\int F(t + n\omega) \, dn = \theta(n) \tag{328}$$

Now, by (260), we have

$$\theta(n) = {}^1F_n + \frac{1}{24} A'_n - \frac{17}{5760} A_n'' + \frac{367}{967680} A_n''' - \dots$$

and hence the preceding equation becomes

$$\int F(t+n\omega) dn = {}^1F_n + \frac{1}{24} A'_n - \frac{17}{5760} A_n'' + \frac{367}{967680} A_n''' - \dots \quad (328a)$$

For brevity, let us put

$$a = +\frac{1}{24} \quad b = -\frac{17}{5760} \quad c = +\frac{367}{967680} \quad \dots \quad (329)$$

and (328a) may be written

$$\int F(t+n\omega) dn = \int F_n dn = {}^1F_n + aA'_n + bA_n'' + cA_n''' + \dots \quad (330)$$

the constant of integration being contained in  ${}^1F_n$ . Multiplying this equation by  $dn$ , and integrating, we get

$$\iint F(t+n\omega) dn^2 = \int {}^1F_n dn + a \int A'_n dn + b \int A_n'' dn + c \int A_n''' dn + \dots \quad (331)$$

Applying formula (330) successively to each of the integrals expressed in the second member of (331), we obtain

$$\begin{aligned} \iint F(t+n\omega) dn^2 &= {}^{11}F_n + aF_n + bA_n'' + cA_n^{iv} + \dots \\ &\quad + a(F_n + aA_n'' + bA_n^{iv} + \dots) \\ &\quad + b(A_n'' + aA_n^{iv} + \dots) \\ &\quad + c(A_n^{iv} + \dots) \\ &\quad + \dots \\ &= {}^{11}F_n + 2aF_n + (a^2 + 2b)A_n'' + 2(ab + c)A_n^{iv} + \dots \end{aligned}$$

Whence, restoring the values of  $a, b, c, \dots$  from (329), and reducing, we obtain

$$\iint F(t+n\omega) dn^2 = {}^{11}F_n + \frac{1}{12} F_n - \frac{1}{240} A_n'' + \frac{31}{60480} A_n^{iv} - \dots \quad (332)$$

If, as in (327), we denote by  $H_0$  the value of  $\int F(T) dT$  which obtains for  $T = t$ , then, by (328), we have

$$H_0 = \left[ \int F(T) dT \right]_{T=t} = \omega \left[ \int F(t+n\omega) dn \right]_{n=0} = \omega \theta(0)$$

and hence, by (272),

$$H_0 = \omega \{ ({}^1F_0) - \frac{1}{12} (A'_0) + \frac{1}{720} (A_0'') - \frac{131}{60480} (A_0''') + \dots \} \quad (333)$$

Upon substituting  $i = 0$  in the first of equations (269), we get

$$({}'F_0) = \frac{1}{2} ({}'F_{-\frac{1}{2}} + {}'F_{\frac{1}{2}}) = {}'F_{\frac{1}{2}} - \frac{1}{2} F_0$$

which, together with (333), gives

$${}'F_{\frac{1}{2}} = \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A_0''') + \frac{1}{60480} (A_0^{iv}) - \dots \quad (334)$$

where the differences enclosed within parentheses are *means* of the corresponding tabular quantities, as defined by (269).

By employing simultaneously the relations (332) and (334), and assigning various limits to the integral, we obtain the following group of formulae :

$$\left. \begin{aligned} {}'F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A_0''') + \frac{1}{60480} (A_0^{iv}) - \dots \\ \int \int_i^{t+\omega} F(T) dT^2 &= \omega^2 \int_0^i F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_i - {}''F_0) + \frac{1}{12} (F_i - F_0) - \frac{1}{240} (A_i'' - A_0'') + \frac{1}{60480} (A_i^{iv} - A_0^{iv}) - \dots \} \end{aligned} \right\} \quad (335)$$

$$\left. \begin{aligned} {}'F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A_0''') + \frac{1}{60480} (A_0^{iv}) - \dots \\ \int \int_i^{t+n\omega} F(T) dT^2 &= \omega^2 \int_0^n F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_n - {}''F_0) + \frac{1}{12} (F_n - F_0) - \frac{1}{240} (A_n'' - A_0'') + \frac{1}{60480} (A_n^{iv} - A_0^{iv}) - \dots \} \end{aligned} \right\} \quad (336)$$

$$\left. \begin{aligned} {}'F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A_0''') + \frac{1}{60480} (A_0^{iv}) - \dots \\ \int \int_{t+n\omega}^{t+\omega} F(T) dT^2 &= \omega^2 \int_n^{t+\omega} F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_i - {}''F_n) + \frac{1}{12} (F_i - F_n) - \frac{1}{240} (A_i'' - A_n'') + \frac{1}{60480} (A_i^{iv} - A_n^{iv}) - \dots \} \end{aligned} \right\} \quad (337)$$

$$\left. \begin{aligned} {}'F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A_0''') + \frac{1}{60480} (A_0^{iv}) - \dots \\ \int \int_{t+n'\omega}^{t+n''\omega} F(T) dT^2 &= \omega^2 \int_{n'}^{n''} F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}''F_{n''} - {}''F_{n'}) + \frac{1}{12} (F_{n''} - F_{n'}) - \frac{1}{240} (A_{n''}'' - A_{n'}'') + \frac{1}{60480} (A_{n''}^{iv} - A_{n'}^{iv}) - \dots \} \end{aligned} \right\} \quad (338)$$

In the preceding group the value of  ${}''F_0$  is wholly arbitrary. We may, however, determine the quantity  ${}''F_0$  such that the sum of the terms in (335) and (336) having the subscript *zero* will vanish: these formulae may therefore be written—

$$\left. \begin{aligned} {}^I F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A'''_0) + \frac{1}{60480} (A^{iv}_0) - \dots \\ {}^{II} F_0 &= -\frac{1}{12} F_0 + \frac{1}{240} A''_0 - \frac{1}{60480} A^{iv}_0 + \dots \\ \int \int_i^{t+i\omega} \bar{F}(T) dT^2 &= \omega^2 \int_0^i \bar{F}(t+n\omega) dn^2 \\ &= \omega^2 ({}^I F_i + \frac{1}{12} F_i - \frac{1}{240} A''_i + \frac{1}{60480} A^{iv}_i - \dots) \end{aligned} \right\} \quad (339)$$

$$\left. \begin{aligned} {}^I F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{12} (A'_0) - \frac{1}{720} (A'''_0) + \frac{1}{60480} (A^{iv}_0) - \dots \\ {}^{II} F_0 &= -\frac{1}{12} F_0 + \frac{1}{240} A''_0 - \frac{1}{60480} A^{iv}_0 + \dots \\ \int \int_i^{t+n\omega} \bar{F}(T) dT^2 &= \omega^2 \int_0^n \bar{F}(t+n\omega) dn^2 \\ &= \omega^2 ({}^I F_n + \frac{1}{12} F_n - \frac{1}{240} A''_n + \frac{1}{60480} A^{iv}_n - \dots) \end{aligned} \right\} \quad (340)$$

Let us now denote the second member of (332) by  $\gamma(n)$ ; that is, let us put

$$\gamma(n) = {}^{II} F_n + \frac{1}{12} F_n - \frac{1}{240} A''_n + \frac{1}{60480} A^{iv}_n - \dots \quad (341)$$

Making  $n = i + \frac{1}{2}$ , this becomes

$$\gamma(i + \frac{1}{2}) = {}^{II} F_{i+\frac{1}{2}} + \frac{1}{12} F_{i+\frac{1}{2}} - \frac{1}{240} A''_{i+\frac{1}{2}} + \frac{1}{60480} A^{iv}_{i+\frac{1}{2}} - \dots \quad (342)$$

It will be observed from the foregoing schedule that  ${}^{II} F_{i+\frac{1}{2}}$ ,  $F_{i+\frac{1}{2}}$ ,  $A''_{i+\frac{1}{2}}$ ,  $\dots$  are not explicitly given, but must be derived from their respective series by interpolation *to halves*. For this purpose, let us put, in analogy with (269),

$$\left. \begin{aligned} ({}^{II} F_{i+\frac{1}{2}}) &= \frac{1}{2} ({}^{II} F_i + {}^{II} F_{i+1}) & (A''_{i+\frac{1}{2}}) &= \frac{1}{2} (A''_i + A''_{i+1}) \\ (F_{i+\frac{1}{2}}) &= \frac{1}{2} (F_i + F_{i+1}) & \dots & \dots \end{aligned} \right\} \quad (343)$$

then, after the manner of (270), we shall have

$$\left. \begin{aligned} {}^{II} F_{i+\frac{1}{2}} &= ({}^{II} F_{i+\frac{1}{2}}) - \frac{1}{8} (F_{i+\frac{1}{2}}) + \frac{3}{128} (A''_{i+\frac{1}{2}}) - \frac{5}{1024} (A^{iv}_{i+\frac{1}{2}}) + \dots \\ F_{i+\frac{1}{2}} &= (F_{i+\frac{1}{2}}) - \frac{1}{8} (A''_{i+\frac{1}{2}}) + \frac{3}{128} (A^{iv}_{i+\frac{1}{2}}) - \dots \\ A''_{i+\frac{1}{2}} &= (A''_{i+\frac{1}{2}}) - \frac{1}{8} (A^{iv}_{i+\frac{1}{2}}) + \dots \\ A^{iv}_{i+\frac{1}{2}} &= (A^{iv}_{i+\frac{1}{2}}) - \dots \\ \dots & \dots \end{aligned} \right\} \quad (344)$$

Upon substituting these expressions in the second member of (342), and reducing, we find

$$\gamma(i + \frac{1}{2}) = ({}^{II} F_{i+\frac{1}{2}}) - \frac{1}{24} (F_{i+\frac{1}{2}}) + \frac{1}{1920} (A''_{i+\frac{1}{2}}) - \frac{3}{193536} (A^{iv}_{i+\frac{1}{2}}) + \dots \quad (345)$$

Again, by means of (332) and (341), we derive

$$\int \int_{n'} F''(t+n\omega) dn^2 = \gamma(n'') - \gamma(n') \quad (346)$$

Finally, denoting by  $H_{-\frac{1}{2}}$  the value of  $\int F(T) dT$  when  $T = t - \frac{1}{2}\omega$ , we shall have, by (328a),

$$\begin{aligned} H_{-\frac{1}{2}} &= \left[ \int F(T) dT \right]_{T=t-\frac{1}{2}\omega} = \omega \left[ \int F(t+n\omega) dn \right]_{n=-\frac{1}{2}} \\ &= \omega \left( {}^I F_{-\frac{1}{2}} + \frac{1}{24} A'_{-\frac{1}{2}} - \frac{1}{5760} A'''_{-\frac{1}{2}} + \frac{3}{967680} A^v_{-\frac{1}{2}} - \dots \right) \end{aligned}$$

which gives

$${}^I F_{-\frac{1}{2}} = \frac{H_{-\frac{1}{2}}}{\omega} - \frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{3}{967680} A^v_{-\frac{1}{2}} + \dots \quad (347)$$

By assigning various values to the limits  $n'$  and  $n''$  in (346), and employing either (341) or (345) as required in each particular case; and finally, by using either (334) or (347) to determine the series  $'F$ , according as the assigned lower limit *is not* or *is* equal to  $-\frac{1}{2}$ , we derive the group of formulae given below:

$$\left. \begin{aligned} {}^I F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{24} (A'_0) - \frac{1}{720} (A'''_0) + \frac{1}{60480} (A^v_0) - \dots \\ {}^{II} F_0 &= -\frac{1}{24} F_0 + \frac{1}{240} A'_0 - \frac{1}{60480} A^{iv}_0 + \dots \\ \int \int_t F(T) dT^2 &= \omega^2 \int \int_0 F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}^{II} F_{i+\frac{1}{2}}) - \frac{1}{24} (F_{i+\frac{1}{2}}) + \frac{1}{1920} (A''_{i+\frac{1}{2}}) - \frac{3}{193536} (A^{iv}_{i+\frac{1}{2}}) + \dots \} \end{aligned} \right\} \quad (348)$$

$$\left. \begin{aligned} {}^I F_{\frac{1}{2}} &= \frac{H_0}{\omega} + \frac{1}{2} F_0 + \frac{1}{24} (A'_0) - \frac{1}{720} (A'''_0) + \frac{1}{60480} (A^v_0) - \dots \\ {}^{II} F_0 &= \text{any convenient value; arbitrarily assigned.} \\ \int \int_{t+n\omega} F(T) dT^2 &= \omega^2 \int \int_n F(t+n\omega) dn^2 \\ &= \omega^2 \{ ({}^{II} F_{i+\frac{1}{2}}) - \frac{1}{24} (F_{i+\frac{1}{2}}) + \frac{1}{1920} (A''_{i+\frac{1}{2}}) - \frac{3}{193536} (A^{iv}_{i+\frac{1}{2}}) + \dots \\ &\quad - {}^{II} F_n - \frac{1}{24} F_n + \frac{1}{240} A''_n - \frac{3}{60480} A^{iv}_n + \dots \} \end{aligned} \right\} \quad (349)$$

$$\left. \begin{aligned} {}^I F_{-\frac{1}{2}} &= \frac{H_{-\frac{1}{2}}}{\omega} - \frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{3}{967680} A^v_{-\frac{1}{2}} + \dots \\ {}^{II} F_0 &= \frac{1}{2} {}^I F_{-\frac{1}{2}} + \frac{1}{24} (F_{-\frac{1}{2}}) - \frac{1}{1920} (A''_{-\frac{1}{2}}) + \frac{3}{193536} (A^{iv}_{-\frac{1}{2}}) - \dots \\ \int \int_{t-\frac{1}{2}\omega} F(T) dT^2 &= \omega^2 \int \int_{-\frac{1}{2}} F(t+n\omega) dn^2 \\ &= \omega^2 ({}^{II} F_i + \frac{1}{24} F_i - \frac{1}{240} A''_i + \frac{3}{60480} A^{iv}_i - \dots) \end{aligned} \right\} \quad (350)$$

$$\left. \begin{aligned}
 {}'F_{-\frac{1}{2}} &= \frac{H_{-\frac{1}{2}}}{\omega} - \frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{367}{967680} A^v_{-\frac{1}{2}} + \dots \\
 {}''F_0 &= \frac{1}{2} {}'F_{-\frac{1}{2}} + \frac{1}{24} (F_{-\frac{1}{2}}) - \frac{1}{1920} (A''_{-\frac{1}{2}}) + \frac{367}{193536} (A^{iv}_{-\frac{1}{2}}) - \dots \\
 \int \int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT^2 &= \omega^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn^2 \\
 &= \omega^2 ({}''F_n + \frac{1}{12} F_n - \frac{1}{240} A''_n + \frac{31}{60480} A^{iv}_n - \dots)
 \end{aligned} \right\} \quad (351)$$

$$\left. \begin{aligned}
 {}'F_{-\frac{1}{2}} &= \frac{H_{-\frac{1}{2}}}{\omega} - \frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{367}{967680} A^v_{-\frac{1}{2}} + \dots \\
 {}''F_0 &= \text{any convenient value; arbitrarily assigned.} \\
 \int \int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT^2 &= \omega^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn^2 \\
 &= \omega^2 \left[ \{({}''F_{i+\frac{1}{2}}) - ({}''F_{-\frac{1}{2}})\} - \frac{1}{24} \{(F_{i+\frac{1}{2}}) - (F_{-\frac{1}{2}})\} \right. \\
 &\quad \left. + \frac{1}{1920} \{(A''_{i+\frac{1}{2}}) - (A''_{-\frac{1}{2}})\} - \frac{367}{193536} \{(A^{iv}_{i+\frac{1}{2}}) - (A^{iv}_{-\frac{1}{2}})\} + \dots \right]
 \end{aligned} \right\} \quad (352)$$

The last formula may also be written in the following form :

$$\left. \begin{aligned}
 {}'F_{-\frac{1}{2}} &= \frac{H_{-\frac{1}{2}}}{\omega} - \frac{1}{24} A'_{-\frac{1}{2}} + \frac{1}{5760} A'''_{-\frac{1}{2}} - \frac{367}{967680} A^v_{-\frac{1}{2}} + \dots \\
 {}''F_0 &= \frac{1}{2} {}'F_{-\frac{1}{2}} + \frac{1}{24} (F_{-\frac{1}{2}}) - \frac{1}{1920} (A''_{-\frac{1}{2}}) + \frac{367}{193536} (A^{iv}_{-\frac{1}{2}}) - \dots \\
 \int \int_{t-\frac{1}{2}\omega}^{t+\frac{1}{2}\omega} F(T) dT^2 &= \omega^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t+n\omega) dn^2 \\
 &= \omega^2 \{({}''F_{i+\frac{1}{2}}) - \frac{1}{24} (F_{i+\frac{1}{2}}) + \frac{1}{1920} (A''_{i+\frac{1}{2}}) - \frac{367}{193536} (A^{iv}_{i+\frac{1}{2}}) + \dots\}
 \end{aligned} \right\} \quad (353)$$

It may be well to again point out the fact that the functions and differences enclosed within parentheses denote the *means* of corresponding tabular quantities, as defined by (269) and (343). Further, that  $H_0$  and  $H_{-\frac{1}{2}}$  denote the values of the *first* integral of  $F(T)$  when for  $T$  we substitute  $t$  and  $t - \frac{1}{2}\omega$ , respectively. Finally, we may add that if in any case  $H_p$  is given and  $H_q$  required, it is only necessary to compute

$$\left. \begin{aligned}
 X &= \int_{t+q\omega}^{t+p\omega} F(T) dT = H_p - H_q \\
 \text{and thence find} \\
 H_q &= H_p - X
 \end{aligned} \right\} \quad (354)$$

In the process of double integration by mechanical quadrature it is sometimes convenient to tabulate, not the given function, but  $\omega^2$  times that quantity. By this means all differences are multiplied by  $\omega^2$ , and thus the *final* multiplication by that factor is avoided. However, in order that the quantities  $'F$  and  $''F$  shall be multiplied by the same factor, it is evident that the independent term  $\frac{H}{\omega}$  (which has the

same fixed value whether we tabulate  $F(T)$  or  $\omega^2 F(T)$  must likewise be multiplied by  $\omega^2$ : so that, proceeding by this method, it becomes necessary to take  $\omega H$  in place of the term  $\frac{H}{\omega}$  which occurs in all the preceding formulae. The computer is cautioned against neglecting this precept in case he tabulates  $\omega^2 F(T)$  instead of the given function  $F(T)$ .

We close the chapter with several examples which illustrate the formulae given above.

EXAMPLE I.—Find the value of

$$Y = - \int_{2.2}^{2.6} \frac{2TdT^2}{(1+T^2)^2}$$

on the supposition that the *first* integral vanishes for  $T = 2.2$ .

We tabulate the given function as below :

$T$	$''F$	$'F$	$F(T) \equiv \frac{-2T}{(1+T^2)^2}$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
2.0			-0.160000				
2.1			0.143501	+16499			
2.2	0.000000		0.129011	14490	-2009		
2.3	-0.063375	-0.063375	0.116267	12744	1746	+263	-32
2.4	0.243017	0.179642	0.105038	11229	1515	231	32
2.5	0.527697	0.284680	0.095125	9913	1316	199	24
2.6	-0.907502	-0.379805	0.086353	8772	1141	175	28
2.7			0.078575	7778	994	147	-17
2.8			-0.071661	+ 6914	- 864	+130	

Here we have

$$t = 2.2 \quad \omega = 0.1 \quad i = 4 \quad H_0 = 0$$

whence, employing (335), we find

$$\begin{array}{rcl}
 F_0 & = & -0.129011 \\
 (\Delta'_0) & = & + \quad 13617 \\
 (\Delta''_0) & = & + \quad 247 \\
 \hline
 \therefore 'F_{\frac{1}{2}} & = & -0.063375
 \end{array}
 \quad
 \begin{array}{rcl}
 + \frac{1}{2} F_0 & = & -0.0645055 \\
 + \frac{1}{12} (\Delta'_0) & = & + \quad 1134.7 \\
 - \frac{1}{720} (\Delta''_0) & = & - \quad 3.8
 \end{array}$$

Assuming  $''F_0 = 0$ , we complete the table as shown above ; thence, proceeding by (335), we obtain

${}''F_4 = -0.907502$	${}''F_0 = 0.000000$	$({}''F_4 - {}''F_0) = -0.907502$
$F_4 = -0.086353$	$F_0 = -0.129011$	$+ \frac{1}{1\frac{1}{2}} (F_4 - F_0) = + 3554.8$
$A_4' = -994$	$A_0' = -1746$	$- \frac{1}{2\frac{1}{4}\ 0} (A_4' - A_0') = - 3.1$
		$\Sigma = -0.903950$
		$\therefore Y = -0.00903950$

*Verification :* Integrating directly, we have

$$\int \frac{-2TdT}{(1+T^2)^2} = \frac{1}{1+T^2} + C$$
$$Y = \left[ \tan^{-1}T + CT \right]_{2.2}^{2.6}$$

whence

$$0 = H_0 = \left[ (1+T^2)^{-1} \right]_{T=2.2} + C$$
$$\therefore C = -0.17123288$$

Finally, using the relation

$$\tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

the preceding expression for  $Y$  becomes

$$Y = \tan^{-1}\left(\frac{0.4}{6.72}\right) + 0.4C$$

which gives

$$Y = -0.00903949$$

EXAMPLE II.—From the table of the preceding example, compute

$$Y = - \int\limits_{2.23}^{2.55} \frac{2TdT^2}{(1+T^2)^2}$$

Here we employ (349), taking

$$t = 2.2 \qquad i = 3 \qquad H_0 = 0 \qquad n = (2.23-2.2) \div 0.1 = 0.30$$

Thus we find

$\dots\dots\dots$	$({}''F_{3\frac{1}{2}}) = -0.717599.5$
$(F_{3\frac{1}{2}}) = -0.090739$	$- \frac{1}{2\frac{1}{4}} (F_{3\frac{1}{2}}) = + 3780.8$
$(A_{3\frac{1}{2}}') = -1068$	$+ \frac{1}{1\frac{7}{8}\ 0} (A_{3\frac{1}{2}}') = - 9.5$
$\Sigma_1 = -0.713828.2$	

Also

$(n = 0.30)$   
 $F_n = -0.125016$   
 $\Delta''_n = -1673$

$-''F_n = +0.006077.9$   
 $-\frac{1}{2}F_n = +0.010418.0$   
 $+\frac{1}{240}\Delta''_n = -7.0$   

---

 $\Sigma_2 = +0.016488.9$   
 $\therefore \Sigma_1 + \Sigma_2 = -0.697339$

whence

$Y = -0.00697339$

Verifying this result as in the preceding example, we find

$Y = \tan^{-1}\left(\frac{0.32}{6.6865}\right) + 0.32C = -0.00697338$

EXAMPLE III.—Let it be required to find

$$Y = -\int\limits_{30^{\circ}}^{50^{\circ}} \frac{M \cos TdT^2}{\sin^2 T}$$

assuming that the *first integral* =  $2M$  when  $T = 30^{\circ}$ ;  $M$  being the modulus of the common system of logarithms.

Here we tabulate  $F(T) \equiv -\omega^2 M \cos T \csc^2 T$  for  $T = 20^{\circ}, 24^{\circ}, 28^{\circ}, \dots, 60^{\circ}$ ; thus avoiding the final multiplication by  $\omega^2$ . Since  $\omega = 4^{\circ} = \pi \div 45$ , we find

$\log \omega^2 M = 7.325659 - 10$

Our table is therefore as follows :

$T$	$''F$	$'F$	$F(T) \equiv$ $-\omega^2 M \cos T \csc^2 T$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$20^{\circ}$			-0.017004				
24			0.011689	+5315			
28			0.008480	3209	-2106	+985	-517
32	+0.029974	+0.060553	0.006392	2088	1121	468	217
36	0.084135	0.054161	0.004957	1435	653	251	113
40	0.133339	0.049204	0.003924	1033	402	138	53
44	0.178619	0.045280	0.003155	769	264	85	30
48	0.220744	0.042125	0.002565	590	179	55	20
52	+0.260304	+0.039560	0.002099	466	124	35	-12
56			0.001722	377	89	+23	
60			-0.001411	+311	-66		

We proceed by formula (353), taking as our data

$$\begin{aligned} t &= 32^\circ & \omega &= 4^\circ = \pi \div 45 \\ i &= 4 & H_{-\frac{1}{2}} &= 2M = 0.868589 \end{aligned}$$

Whence, observing that we must now take  $\omega H_{-\frac{1}{2}}$  instead of the term  $H_{-\frac{1}{2}} \div \omega$  in (353), the computation of  $'F_{-\frac{1}{2}}$  is as follows :

$$\begin{array}{rcl} \log \omega H_{-\frac{1}{2}} & = & 8.782752 \\ A'_{-\frac{1}{2}} & = & + 2088 \\ A''_{-\frac{1}{2}} & = & + 468 \\ \hline & & \omega H_{-\frac{1}{2}} = +0.060639.0 \\ & & -\frac{1}{24} A'_{-\frac{1}{2}} = - 87.0 \\ & & + \frac{1}{5760} A''_{-\frac{1}{2}} = + 1.4 \\ \hline & & \therefore 'F_{-\frac{1}{2}} = +0.060553.4 \end{array}$$

And for  $''F_0$  we find

$$\begin{array}{rcl} . . . . . & & \frac{1}{2} 'F_{-\frac{1}{2}} = +0.030276.7 \\ (F_{-\frac{1}{2}}) & = & -0.007436 \\ & & + \frac{1}{24} (F_{-\frac{1}{2}}) = - 309.8 \\ (A''_{-\frac{1}{2}}) & = & - 887 \\ & & - \frac{1}{1920} (A''_{-\frac{1}{2}}) = + 7.9 \\ (A^{iv}_{-\frac{1}{2}}) & = & - 367 \\ & & + \frac{1}{19360} (A^{iv}_{-\frac{1}{2}}) = - 0.7 \\ \hline & & \therefore ''F_0 = +0.029974 \end{array}$$

Upon completing the table as shown above, and continuing the computation by (353), we obtain

$$\begin{array}{rcl} (i = 4) & & ({}''F_{\frac{4}{3}}) = +0.240524.0 \\ (F_{\frac{4}{3}}) & = & -0.002332 \\ & & - \frac{1}{24} (F_{\frac{4}{3}}) = + 97.2 \\ (A''_{\frac{4}{3}}) & = & - 106 \\ & & + \frac{1}{1920} (A''_{\frac{4}{3}}) = - 0.9 \\ \hline & & \therefore Y = +0.240620 \end{array}$$

We easily verify this result analytically as follows :

$$\begin{aligned} \int \frac{-M \cos T dT}{\sin^2 T} &= \frac{M}{\sin T} + C \\ \iint \frac{-M \cos T dT^2}{\sin^2 T} &= M \log_e \tan \frac{1}{2} T + CT + C' \\ &= \log_{10} \tan \frac{1}{2} T + CT + C' \\ \therefore Y &= \left[ \log_{10} \tan \frac{1}{2} T + CT \right]_{T=30^\circ=\frac{\pi}{6}}^{T=50^\circ=\frac{5}{8}\pi} \end{aligned}$$

But

$$\begin{aligned} 2M &= H_{-\frac{1}{2}} = \frac{M}{\sin 30^\circ} + C = 2M + C \\ \therefore C &= 0 \\ \therefore Y &= \log_{10} \tan \left( \frac{50^\circ}{2} \right) - \log_{10} \tan \left( \frac{30^\circ}{2} \right) \end{aligned}$$

Now we find

$$\begin{array}{r} \log \tan 25^\circ = 9.668672.5 - 10 \\ \log \tan 15^\circ = 9.428052.5 - 10 \\ \hline \therefore Y = 0.240620 \end{array}$$

which agrees exactly with the former result.

EXAMPLE IV.—From the table and data of Example III, compute the integral

$$Y = - \int_{30^\circ}^{45^\circ} \frac{M \cos T dT}{\sin^2 T}$$

Here we employ (351), taking  $t = 32^\circ$  as before; we then have for the value of  $n$  at the upper limit,

$$n = (45^\circ - 32^\circ) \div 4^\circ = 3.25 = 3 + 0.25$$

We therefore obtain

$$\begin{array}{rcl} . & . & . \\ F_n = -0.002993 & + \frac{1}{12} F_n = - & 249.4 \\ A_n'' = -163 & - \frac{1}{240} A_n'' = + & 0.7 \\ \hline & \therefore Y = & +0.189172 \end{array}$$

Verifying this result as in the last example, we find

$$Y = \log_{10} \tan 22^\circ 30' - \log_{10} \tan 15^\circ = +0.189172$$

EXAMPLE V.—As a final exercise, combining both single and double integration, and illustrating, moreover, the use of formula (339) when several values are assigned in succession to the integer  $i$ , we shall conclude these examples with a complete and detailed solution of the following problem:

A particle  $P$  of unit mass is impelled along a straight line  $AB$  by a varying force whose expression is  $20000T^{-3}$ ; where  $T$  is the time in seconds after a definite epoch, and the implied unit of length is one foot. It is required to find by quadratures the velocity,  $v$ , and the distance,  $AP = x$ , for the times

$$T = 102, 104, 106, 108 \text{ and } 110 \text{ seconds, respectively;}$$

assuming that  $v_0 = 0.6$  feet per second and  $x_0 = 8$  feet when  $T_0 = 100$  seconds.

Since the mass of  $P$  is unity, we have, simply,

$$\frac{d^2x}{dT^2} = \frac{20000}{T^3}$$

whence by a single integration

$$v = \frac{dx}{dT} = \int_{T_0}^T \frac{20000dT}{T^3} + v_0 \tag{\alpha}$$

and by double integration

$$x = \iint_{T_0}^T \frac{20000dT^2}{T^3} + x_0 \tag{\beta}$$

We shall first compute the required values of  $x$  as given by equation  $(\beta)$ , effecting the double integration by means of (339). The details of the computation are shown in the following table :

TABLE (A).

$T$	$F(T) \equiv \frac{20000}{T^3}$	$\Delta'$	$\Delta''$	$'F$	$''F + \frac{1}{2}x_0 \equiv a$	$+\frac{1}{12}F \equiv b$	$\frac{1}{2}x = a + b$	$x$
96	0.04521							
98	.04250	−271	+21	+0.53730				
100	.04000	250	19	.57980	+3.99667	+0.00333	4.00000	8.0000
102	.03769	231	18	.61980	4.61647	314	4.61961	9.2392
104	.03556	213	15	.65749	5.27396	296	5.27692	10.5538
106	.03358	198	15	.69305	5.96701	280	5.96981	11.9396
108	.03175	183	13	.72663	6.69364	265	6.69629	13.3926
110	.03005	170	12	.75838	+7.45202	+0.00250	7.45452	14.9090
112	.02847	158		.78843				
114	0.02700	−147	+11	+0.81690				

Since we shall afterwards use this same table in finding  $v$  by single integration, it is here convenient to tabulate  $\omega$  times the given function : thus avoiding the final multiplication by  $\omega$  in computing  $v$ , and reducing the corresponding factor in the case of  $x$  from  $\omega^2$  to  $\omega$ . Accordingly, we tabulate under  $F(T)$  the function

$$F(T) \equiv 20000\omega T^{-3} = 40000 T^{-3}$$

Assume  $t = 100$ , and proceed by (339). To determine  $'F_{\frac{1}{2}}$ , it must be observed that since  $F(T), \Delta', \Delta'', \dots$  already contain the factor  $\omega$ , it is here necessary to multiply the independent term  $\frac{H_0}{\omega}$

by the same factor: so that, writing  $v_0 (= H_0)$  for  $\frac{H_0}{\omega}$  in the first equation of (339), and omitting insensible terms, we have

$${}'F_{\frac{1}{2}} = v_0 + \frac{1}{2} F_0 + \frac{1}{1\frac{1}{2}} (A'_0) \quad (\gamma)$$

Hence, substituting  $v_0 = 0.6$ ,  $F_0 = 0.04000$ ,  $(A'_0) = \frac{1}{2} (A'_{-\frac{1}{2}} + A'_{\frac{1}{2}}) = -0.00240$ , we find  ${}'F_{\frac{1}{2}} = +0.61980$ , and thus complete the series  ${}'F$  as given above.

The second equation of (339) gives simply,  ${}''F_0 = -\frac{1}{1\frac{1}{2}} F_0$ , the term in  $A''$  being insensible. But since, by equation  $(\beta)$ , we should afterwards have to add the constant  $x_0$  to each computed value of the double integral taken from  $T_0$  to  $T$ , it is expedient to tabulate in place of  ${}''F_0$  the quantity

$${}''F_0 + \frac{x_0}{\omega} = {}''F_0 + \frac{1}{2} x_0 = -\frac{1}{1\frac{1}{2}} F_0 + 4.0 = 4.0 - 0.00333 = +3.99667$$

and thence complete the series as given under  ${}''F + \frac{1}{2} x_0 \equiv a$ . The reason for this procedure is easily made apparent: for the final equation of (339) gives (since  $\omega^2$  must now be replaced by  $\omega$ )

$$\iint_{T_0}^T \frac{20000 dT^2}{T^3} = \omega ({}''F_i + \frac{1}{1\frac{1}{2}} F_i)$$

and substituting this expression in equation  $(\beta)$ , we obtain

$$x = \omega ({}''F_i + \frac{1}{1\frac{1}{2}} F_i) + x_0 = \omega ({}''F_i + \frac{x_0}{\omega} + \frac{1}{1\frac{1}{2}} F_i) \quad (\delta)$$

Therefore, upon forming the column  $+\frac{1}{1\frac{1}{2}} F \equiv b$ , as given above, we have from  $(\delta)$

$$\frac{1}{2} x = {}''F_i + \frac{1}{2} x_0 + \frac{1}{1\frac{1}{2}} F_i = a + b$$

whence the required values of  $x$  are derived and tabulated in the final column of Table (A).

For the computation of the velocity  $v$  we employ formula (282), the first equation of which gives

$${}'F_{-\frac{1}{2}} = -\frac{1}{2} F_0 + \frac{1}{1\frac{1}{2}} (\Delta'{}_0)$$

or, by adding  $F_0$  to both members,

$${}'F_{\frac{1}{2}} = +\frac{1}{2} F_0 + \frac{1}{1\frac{1}{2}} (\Delta'{}_0)$$

But we shall avoid subsequent additions of the constant  $v_0$ , required by equation (a), if we increase this value of  ${}'F_{\frac{1}{2}}$  by the term  $v_0 = 0.6$ ; that is, if we take

$${}'F_{\frac{1}{2}} = v_0 + \frac{1}{2} F_0 + \frac{1}{1\frac{1}{2}} (\Delta'{}_0)$$

which is the same as the expression ( $\gamma$ ), used for determining the series  ${}'F$  in Table (A). The latter series is therefore to be employed in finding  $v$ , the computation of which is as follows:

TABLE (B).

$T$	$({}'F)$	$(\Delta')$	$-\frac{1}{1\frac{1}{2}} (\Delta')$	$v = ({}'F) - \frac{1}{1\frac{1}{2}} (\Delta')$
96	+0.51470	. .	+24	+0.51494
98	.55855	-260	22	.55877
100	.59980	240	20	.60000
102	.63865	222	18	.63883
104	.67527	205	17	.67544
106	.70984	190	16	.71000
108	.74251	176	15	.74266
110	.77341	164	14	.77355
112	.80267	-152	13	.80280
114	+0.83040	. .	+12	+0.83052

Recalling the fact that functions and differences in parentheses are *means* taken according to (269), the method of forming the second, third and fourth columns of this table from the quantities of Table (A) is obvious. Now, since the factor  $\omega$  has been previously introduced, the second equation of (282) gives

$$v = ({}'F_i) - \frac{1}{1\frac{1}{2}} (\Delta'_i)$$

from which expression the required values of  $v$  are computed and tabulated in the final column of Table (B).

This completes the solution of the problem. An interesting check is derived, however, by observing that equation (a) gives

$$x = \int_{x_0}^T v dT + x_0 \quad (\epsilon)$$

whence  $x$  may be obtained from the series  $v$  by single integration. For this purpose we make  $f(T) \equiv \omega v = 2v$ , and thus form the table below :

TABLE (C).

$T$	$f(T) \equiv 2v$	$\delta'$	$\delta''$	$'f + x_0$	$('f) + x_0 \equiv c$	$(\delta')$	$-\frac{1}{12}(\delta') \equiv d$	$x = c + d$
96	1.0299	+876						
98	1.1175	825	-51	+ 7.4067				
100	1.2000	777	48	8.6067	8.0067	+801	-67	8.0000
102	1.2777	732	45	9.8844	9.2455	754	63	9.2392
104	1.3509	691	41	11.2353	10.5598	711	59	10.5539
106	1.4200	653	38	12.6553	11.9453	672	56	11.9397
108	1.4853	618	35	14.1406	13.3979	636	53	13.3926
110	1.5471	585	33	15.6877	14.9141	+602	-50	14.9091
112	1.6056	+554	-31					
114	1.6610							

Here again we take  $t = 100$ , and employ (282), which gives

$$'f_{-4} = -\frac{1}{2}f_0 + \frac{1}{12}(\delta'_0) = -0.6000 + 0.0067 = -0.5933$$

Increasing this value by  $x_0 = 8.0$ , to provide for the constant  $x_0$  in equation ( $\epsilon$ ), we get +7.4067, which number is written under  $'f + x_0$ , on the line  $t - \frac{1}{2}\omega$ . Completing this column by successive additions of the functions  $f$ , we next form the series of *mean* values tabulated under  $('f) + x_0 \equiv c$ . The columns  $(\delta')$  and  $-\frac{1}{12}(\delta') \equiv d$  are then computed, and finally the column  $x = c + d$ . These values of  $x$  agree substantially with those given in Table (A).

From the given analytical expression for the force, together with the initial conditions of the problem, we easily find

$$v = 1.6 - 10000T^{-2} \quad , \quad x = 1.6T + 10000T^{-1} - 252$$

whence, making  $T = 110$ , we obtain

$$v = 0.77355 \quad \text{and} \quad x = 14.9091$$

which further verify the results derived by quadratures.

79. It is worth while to inquire what change takes place in the value of the double integral

$$Y = \iint_{T'}^{T''} F(T) dT^2$$

when, in a particular problem, the quantity  $H$  is changed from an assigned value  $H'$  to a new value  $H''$ . This is easily answered. For, if we change  $H'$  to  $H''$ , the value of the first integral—corresponding to *any* particular value of  $T$ —is thereby increased by the quantity  $H''-H'$ ; or, what amounts to the same thing, the *constant* of the first integration,  $M$  in (286a), is thus increased by  $H''-H'$ . Therefore, by (287), it is evident that  $Y$  is increased by the quantity  $(H''-H')(T''-T')$ .

## EXAMPLES.

1. Given the semi-major axis of an ellipse,  $a = 1$ , and the semi-minor axis,  $b = 0.8$ , to find the length of the elliptic quadrant.

*Ans.* 1.41808.

[NOTE: — Take the eccentric angle  $E$  as independent variable, and hence find

$$s = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \cos^2 E} dE$$

where  $e$  is the eccentricity, and  $s$  the required length.]

2. Given the equation of a cardioid,  $r = 1 + \cos \theta$ : to find, by mechanical quadrature, the length of that part of the curve comprised between the initial line and a line through the pole at right-angles to the initial line.

*Ans.* 2.82843.

3. The equation of a curve being  $y = x^2 \sqrt{2 - \sin x}$ , find the area included between the curve, the axis of  $x$ , and the two ordinates,  $x = \frac{\pi}{5}$  and  $x = \frac{2}{7}\pi$ .

*Ans.* 0.180518.

4. Compute the value of

$$Y = \iint_0^{\frac{\pi}{8}} \frac{dT^2}{\sqrt{1 - 0.82 \sin^2 T}}$$

assuming that the *first* integral vanishes at the lower limit.

*Ans.* 0.139727.

5. Given a curve in a vertical plane whose points satisfy the relation

$$\frac{d^2y}{dx^2} = \frac{4x^2 - 3}{5 + \sqrt{x}}$$

—the axis of  $y$  being vertical. Find the difference of level between two points whose abscissae are 1.000 and 1.473, respectively; assuming the direction of the curve to be horizontal at the first point.

*Ans.* 0.044228.

6. By what amount would the preceding result be changed by supposing the tangent to the curve at the first point to be inclined  $45^\circ$  to the horizontal?

[NOTE: — This question should be answered mentally.]

# CHAPTER V.

## MISCELLANEOUS PROBLEMS AND APPLICATIONS.

80. The present short chapter will be devoted to the solution of a number of problems and examples involving certain principles and precepts hitherto established.

81. PROBLEM I.—*To find  $S \equiv 1^k + 2^k + 3^k + \dots + r^k$ , where  $k$  and  $r$  are integers.*

The method of solution is best illustrated by assigning a particular value to  $k$ . Thus, let it be required to find

$$S \equiv 1^4 + 2^4 + 3^4 + \dots + r^4$$

We tabulate below and difference the values of  $T^4$  which correspond to  $T = 1, 2, 3, 4, 5$  and  $6$ . Thus we find :

$T$	${}^1F$	$F(T) \equiv T^4$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
1	${}^1F_0$	1					
2	${}^1F_1$	16	15	50			
3	${}^1F_2$	81	65	110	60	24	
4	.	256	175	194	84	24	0
5	.	625	369	302	108	.	.
6	.	1296	671	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$r-1$	${}^1F_{r-1}$	$(r-1)^4$	.	.	.	.	.
$r$	${}^1F_r$	$r^4$	.	.	.	.	.

Now, by Theorem V, the 4th differences of  $F(T)$  are constant, and hence the 5th and higher differences all *vanish*. Whence, if we

consider the auxiliary series  $'F$ —defined as in Chapter IV—we shall have, by the fundamental formula (73),

$$\begin{aligned} 'F_r &= 'F_0 + r + \frac{r(r-1)}{1^2}(15) + \frac{r(r-1)(r-2)}{1^3}(50) \\ &\quad + \frac{r(r-1)(r-2)(r-3)}{1^4}(60) + \frac{r(r-1)\dots(r-4)}{1^5}(24) \\ &= 'F_0 + \frac{r}{30}(r+1)(2r+1)(3r^2+3r-1) \end{aligned}$$

Therefore, by Theorem I, we have

$$S = 'F_r - 'F_0 = \frac{r}{30}(r+1)(2r+1)(3r^2+3r-1) \quad (355)$$

which is the required expression for the sum of the fourth powers of the first  $r$  integers.

82. PROBLEM II.—*Given a series of functions,  $F_{-3}, F_{-2}, F_{-1}, F_0, F_1, F_2, \dots$ , and an assigned intermediate value,  $F_n$ : To find the corresponding interval  $n$ .*

*First Solution:* The simplest method is to determine by inspection an *approximate* value of  $n$ , and then find by direct interpolation the values of the function corresponding to three or four closely equidistant values of  $n$  that shall embrace the required interval. The latter is then readily found by a simple interpolation.

EXAMPLE.—From the following ephemeris find the time when the logarithm of *Mercury's* distance from the Earth = 9.7968280: that is, given  $F_n = 9.7968280$ , to find  $n$ . The tabular quantities are here given for every second Greenwich mean noon.

Date 1898	Log. Dist. of ☿ from ☉	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$	$\Delta^v$
May 8	9.7560706	+ 91669				
10	9.7652375	116508	+ 24839			
12	9.7768883	136599	20091	− 4748	+ 382	
14	9.7905482	152324	15725	4366	457	+ 75
16	9.8057806	164140	11816	3909	+ 592	+ 135
18	9.8221946	+ 172639	+ 8499	− 3317		
20	9.8394585					

We observe that the given logarithm falls somewhere between the tabular values for May 14 and 16, and soon find that the interval

(from the former date) is somewhat greater than 0.4. Hence we take  $F_0 = 9.7905482$ , and interpolate — by BESSEL'S Formula — the functions corresponding to  $n = 0.38, 0.41$ , and  $0.44$ . Thus, computing and differencing these values, we find

$n$	$F_n$	$\Delta'$	$\Delta''$
0.38	9.7961736		
0.41	9.7966267	+4531	
0.44	9.7970810	+4543	+12

Whence, if we denote by  $n'$  the interval at which the required function lies beyond the middle function in this new series, we shall have, by neglecting the small second difference,

$$n' = 2013 \div 4543 = 0.44, \text{ nearly.}$$

But if great accuracy is required, we may easily take account of the second difference by the method of the *corrected first difference* (§44). Thus, in the last table, we find that the corrected first difference which corresponds to  $n' = 0.44$  is 4540; hence we have

$$\begin{aligned} n' &= 2013 \div 4540 = 0.4434 \\ \therefore n &= 0.41 + 0.4434 \times 0.03 = 0.423302 \end{aligned}$$

The required time is, therefore,

$$T = \text{May } 14^{\text{d}} + 0.423302 \times 48^{\text{h}} = \text{May } 14^{\text{d}} \ 20^{\text{h}} \ 19^{\text{m}} \ 6^{\text{s}}.6$$

83. *Second Solution of Problem II.* — Given  $F_n$ , to find the value of  $n$ .

Let  $m$  denote an approximate value of  $n$ , true to the *nearest tenth* of a unit, and put

$$n = m + z \tag{356}$$

Then we have

$$\begin{aligned} F_n &= F_{m+z} = F[t + (m+z)\omega] = F[(t+m\omega) + z\omega] \\ &= F(t+m\omega) + z\omega F'(t+m\omega) + \frac{z^2\omega^2}{2} F''(t+m\omega) + \dots \end{aligned}$$

Since we have supposed  $z$  not to exceed 0.05, it is permissible to neglect  $z^3, z^4, \dots$  in the last expression, which becomes, therefore,

$$F_n = F_m + z\omega F'_m + \frac{1}{2} z^2 \omega^2 F''_m \tag{357}$$

To find  $z$  from this equation, we first neglect the small term in  $z^2$ , and thus obtain an approximate value which we shall call  $x$ . In this manner we find

$$x = \frac{F_n - F_m}{\omega F'_m} \quad (358)$$

This approximate value of  $z$  will now suffice for substitution in the last term of (357). Accordingly, we obtain

$$z = x - \frac{1}{2}x^2 \left( \frac{\omega^2 F''_m}{\omega F'_m} \right) \quad (359)$$

whence, putting

$$y = \frac{1}{2}x^2 \left( \frac{\omega^2 F''_m}{\omega F'_m} \right) \quad (360)$$

we have

$$z = x - y$$

and equation (356) becomes

$$n = m + x - y \quad (361)$$

Finally, to express  $F_m$ ,  $\omega F'_m$ , and  $\omega^2 F''_m$  in terms of the differences of the given series  $F$ , it will be expedient to employ STIRLING'S Formula of interpolation, together with the expressions for  $F'_m$  and  $F''_m$  as developed in §61. The above solution may then be expressed as follows :

Determine	$m = \text{an approximate value of } n, \text{ true to the nearest tenth of a unit.}$	}	(362)
Thence find	$F_m = F_0 + ma + Bb_0 + Cc + Dd_0 + \dots$		
	$D_1 \equiv \omega F'_m = a + mb_0 + C'c + D'd_0 + \dots$		
	$D_2 \equiv \omega^2 F''_m = b_0 + mc + \dots$		
	$K = \frac{D_2}{D_1}$		
	$x = \frac{F_n - F_m}{D_1}$		
	$y = \frac{1}{2}x^2 K$		
and	$n = m + x - y$		

Here the differences are to be taken according to the schedule on page 62 ; the coefficients  $B, C, D, \dots$  being taken from Table II, and  $C', D', \dots$  from Table V. Finally, Table VII gives the value of  $y$  for top argument  $K$  and side argument  $x$  ; observing that  $y$  has the same sign as  $K$ .

EXAMPLE.—Same as in §82.

Here we find  $m = 0.40$ ; and hence take from the given table, and from Tables II and V, the quantities

$m = 0.40$	$a = +144461.5$	$\dots\dots\dots$
$B = +0.080$	$b_0 = +15725$	$\dots\dots\dots$
$C = -0.056$	$c = -4137.5$	$C' = -0.08667$
$D = -0.0056$	$d_0 = +457$	$D' = -0.02267$
$E = +0.01075$	$e = +105$	$E' = +0.01440$

The computation of  $F_m$ ,  $D_1$  and  $D_2$  by (362) is therefore as follows:

$F_0 = 9.7905482$	$\dots\dots\dots$	$\dots\dots\dots$
$ma = +57784.6$	$a = +144461.5$	$\dots\dots\dots$
$Bb_0 = +1258.0$	$mb_0 = +6290.0$	$b_0 = +15725$
$Cc = +231.7$	$C'c = +358.6$	$mc = -1655$
$Dd_0 = -2.6$	$D'd_0 = -10.4$	$\dots\dots\dots$
$Ee = +1.1$	$E'e = +1.5$	$\dots\dots\dots$
$\therefore F_m = 9.7964755$	$\therefore D_1 = +151101$	$\therefore D_2 = +14070$
$F_n = 9.7968280$		$\therefore F_n - F_m = +3525$

Whence

$$K = D_2 \div D_1 = +14070 \div 151101 = +0.0931$$

$$x = (F_n - F_m) \div D_1 = +3525 \div 151101 = +0.023329$$

and we finally obtain

$$\begin{aligned} m &= 0.400000 \\ x &= +0.023329 \\ \text{(Table VII) } -y &= -26 \\ \hline \therefore n &= 0.423303 \end{aligned}$$

which agrees within one unit with the former result.

84. PROBLEM III.—*To solve any numerical equation whatever involving but one unknown quantity.*

The given equation, whether simple or complex, algebraic or transcendental, may be written in the form

$$F(T) = 0$$

The problem therefore reduces to the question of finding  $n$  when  $F_n$  is known and equal to zero—which is the same as Problem II.

EXAMPLE.—Solve the transcendental equation

$$T - 20^\circ \sin T = 45^\circ$$

where  $T$  is expressed in degrees of arc.

This equation may be written

$$F(T) \equiv T - 20^\circ \sin T - 45^\circ = 0$$

which by trial we find to be satisfied by a value of  $T$  not far from  $63^\circ$ ; hence we tabulate  $F(T)$  for  $T = 62^\circ, 63^\circ$ , and  $64^\circ$ , as follows:

$T$	$F(T)$	$\Delta'$	$\Delta''$
$62^\circ$	$-0.6590$		
$63^\circ$	$+0.1799$	$+8389$	$+53$
$64^\circ$	$+1.0241$	$+8442$	

Here we have given  $F_n = 0$ , to find  $n$ . Whence, employing the *corrected first difference* (§ 45), we find

$$T = 63^\circ - \frac{1799}{8410} \times 1^\circ = 62.7861$$

85. PROBLEM IV.—*Given a series of numerical functions embracing a maximum or minimum value: To find the value of the argument which corresponds to the maximum or minimum function.*

Find by inspection the tabular function which falls *nearest* the required maximum or minimum value. Call this tabular function  $F_0$ . Then, from the schedule

$T$	$F(T)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
$t - \omega$	$F_{-1}$	$a'$	$b'$		$d'$
$t$	$F_0$		$b_0$	$c'$	$d_0$
$t + \omega$	$F_1$		$b_1$	$c_1$	$d_1$

we have, by the first of equations (182),

$$F'(T) = F'(t+n\omega) \\ = \frac{1}{\omega} \left[ (a - \frac{1}{6}c + \dots) + n(b_0 - \frac{1}{2}d_0 + \dots) + \frac{1}{2}n^2(c - \dots) + \frac{1}{6}n^3(d_0 - \dots) + \dots \right]$$

Therefore, since the condition of maximum or minimum requires that  $F'(T) = 0$ , we have, by neglecting 5th differences,

$$(a - \frac{1}{6}c) + (b_0 - \frac{1}{12}d_0)n + \frac{1}{2}cn^2 + \frac{1}{6}d_0n^3 = 0 \quad (363)$$

which determines the value of  $n$ , and hence, also, the value of  $T$ , at the point of maximum or minimum of  $F(T)$ . This equation may be readily solved by successive approximations, by first neglecting the terms containing  $n^2$  and  $n^3$ , and afterwards substituting therein the approximate value of  $n$  thus found, and so on; or, we may consider the solution of (363) from the standpoint of Problem III,—which may be regarded as the more direct of the two methods.

EXAMPLE.—The following ephemeris gives the log radius vector of *Mars* with respect to the Sun ( $\log r$ ). Find the time of perihelion passage of the planet.

Date 1898	Log $r$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
April 6	0.1416628				
14	0.1409303	—7325	+2844		
22	0.1404822	4481	2891	+47	—27
30	0.1403232	—1590	2911	+20	33
May 8	0.1404553	+1321	2898	—13	—36
16	0.1408772	4219	+2849	—49	
24	0.1415840	+7068			

Here we are required to find the instant when  $\log r$  is a minimum. Since it is evident that this condition occurs only a few hours from April 30, we take  $F_0 = 0.1403232$ . Whence, from the above table, we find

$$\begin{array}{ll} a = -134.5 & a - \frac{1}{6}c = -135 \\ b_0 = +2911 & b_0 - \frac{1}{12}d_0 = +2914 \\ c = +3.5 & \frac{1}{2}c = +2 \\ d_0 = -33 & \frac{1}{6}d_0 = -6 \end{array}$$

and therefore, by (363),

$$-135 + 2914n + 2n^2 - 6n^3 = 0$$

or

$$2914n = 135 - 2n^2 + 6n^3$$

Neglecting the last two terms of this equation, we have, for an approximate value of  $n$ ,

$$n = 135 \div 2914 = 0.046, \text{ nearly};$$

and since for this value of  $n$  the small terms sensibly vanish, we obtain as our final value

$$n = 135 \div 2914 = 0.04633$$

The date of perihelion passage is, therefore,

$$T = \text{April } 30^{\text{d}} + 0.04633 \times 8 \times 24^{\text{h}} = \text{April } 30^{\text{d}} 8^{\text{h}}.895$$

86. PROBLEM V.—*Given a series of numerical values ( $F_{-3}, F_{-2}, F_{-1}, F_0, F_1, F_2, \dots$ ) of any function  $F(T)$  which is analytically unknown: To find an approximate algebraic expression for  $F(T)$  in terms of the variable argument.*

Let us put

$$\tau = T - t \tag{364}$$

and TAYLOR'S Theorem gives

$$F(T) = F(t + \tau) = F(t) + \tau F'(t) + \frac{\tau^2}{2} F''(t) + \frac{\tau^3}{6} F'''(t) + \dots \tag{365}$$

Upon substituting in (365) the expressions for  $F'(t)$ ,  $F''(t)$ ,  $F'''(t)$ ,  $\dots$ , as given by (175), we obtain

$$\begin{aligned} F(T) = F(t) &+ \frac{1}{\omega} (a - \frac{1}{6} c + \frac{1}{36} e - \dots) \tau + \frac{1}{\omega^2 2} (b_0 - \frac{1}{12} d_0 + \dots) \tau^2 \\ &+ \frac{1}{\omega^3 6} (c - \frac{1}{4} e + \dots) \tau^3 + \frac{1}{\omega^4 24} (d_0 - \dots) \tau^4 + \frac{1}{\omega^5 120} (e - \dots) \tau^5 + \dots \end{aligned} \tag{366}$$

which expresses  $F(T)$  as a rational integral function of  $\tau$ , with known numerical coefficients;  $\tau$  being the value of the variable argument counted from the fixed epoch  $t$ , as defined by (364).

EXAMPLE. — From NEWCOMB'S *Astronomical Constants* we take the following table of the mean obliquity of the ecliptic ( $\epsilon$ ) for every fifth century :

Year	Obliquity	$\Delta'$	$\Delta''$	$\Delta'''$
	$^{\circ} \quad ' \quad ''$	$' \quad ''$	$''$	$''$
0	23 41 43.78	—3 45.81		
500	37 57.97	3 49.90	—4.09	+1.35
1000	34 8.07	3 52.64	2.74	1.36
1500	30 15.43	3 54.02	1.38	+1.36
2000	26 21.41	—3 54.04	—0.02	
2500	23 22 27.37			

Let it be required to express  $\epsilon$  in terms of  $\tau$ , the latter being counted from the year 1000 in terms of a century as the unit.

Since we adopt one century as the unit of time, it is necessary to express  $\omega$  in the same unit; therefore we have

$$\begin{aligned} \omega &= 5 & t &= 1000^y & F(t) &= 23^{\circ} 34' 8''.07 \\ a &= -3' 51''.27 = -231''.27 & b_0 &= -2''.74 & c &= +1''.355 \\ a - \frac{1}{6}c &= -231''.496 & \omega^2 \underline{2} &= 50 & \omega^3 \underline{3} &= 750 \end{aligned}$$

Whence, by (366), we obtain

$$\begin{aligned} \text{Coefficient of } \tau &= -231.496 \div 5 = -46.299 \\ \text{“ “ } \tau^2 &= -2.74 \div 50 = -0.0548 \\ \text{“ “ } \tau^3 &= +1.355 \div 750 = +0.00181 \end{aligned}$$

Accordingly, the required expression for the obliquity is—

$$\epsilon = 23^{\circ} 34' 8''.07 - 46''.299 \tau - 0''.0548 \tau^2 + 0''.00181 \tau^3$$

*Verification*: Putting  $\tau = 10$  in this formula, we should get the obliquity for 2000. Now we find

$$(\text{For } 2000) \quad \epsilon = 23^{\circ} 34' 8''.07 - 462''.99 - 5''.48 + 1''.81 = 23^{\circ} 26' 21''.41$$

which agrees exactly with the tabular value above.

It will be observed that the solution given by (366) restricts the *epoch*, or origin from which  $\tau$  is counted, to some *tabular* value of the argument, as  $t$ . Should the assigned epoch be some *intermediate* value of  $T$ , say  $T_1$ , it will only be necessary to write

$$\tau_1 = T - T_1$$

and we have

$$F(T) = F(T_1 + \tau_1) = F(T_1) + \tau_1 F'(T_1) + \frac{\tau_1^2}{2} F''(T_1) + \dots$$

Therefore, if we put

$$\text{we shall have } \left. \begin{aligned} T_1 &= t + m\omega \\ F(T) &= F_m + \tau_1 F'_m + \frac{\tau_1^2}{2} F''_m + \frac{\tau_1^3}{6} F'''_m + \dots \end{aligned} \right\} \quad (366a)$$

where  $\tau_1 (= T - T_1)$  is the value of the variable argument counted from the assigned epoch  $T_1$ . Accordingly, if we compute by the usual methods the values of  $F_m, F'_m, F''_m, F'''_m, \dots$ , and substitute these in (366a), we shall obtain the expression required.

As an example, let us express the obliquity ( $\epsilon$ ) as a function of the time ( $\tau_1$ ) counted from the epoch 1600.0 in terms of a century as the unit.

Reverting to the above table, we take

$$t = 1500^y \qquad T_1 = 1600^y \qquad m = 0.20$$

Whence we find

$$F_m = 23^\circ 29' 28''.69 \qquad F'_m = -46''.761 \qquad F''_m = -0''.0443 \qquad F'''_m = +0''.01088$$

Substituting these values in the formula (366a), we obtain the required expression, namely,

$$\epsilon = 23^\circ 29' 28''.69 - 46''.761 \tau_1 - 0''.0222 \tau_1^2 + 0''.00181 \tau_1^3$$

87. GEOMETRICAL PROBLEM.—A circular well four feet in diameter is centrally intersected by a horizontal cylindrical shaft whose diameter is one foot. Find the volume of the portion of the shaft within the well.

*Solution:* Consider a vertical section or lamina of the shaft parallel to its axis, at a horizontal distance  $x$  from the latter, and having the differential thickness  $dx$ . Then, if we denote the radii of well and shaft by  $R$  and  $r$ , respectively, we shall have for the length of this rectangular section

$$l = 2\sqrt{R^2 - x^2}$$

and for its breadth, or height,

$$h = 2\sqrt{r^2 - x^2}$$

Therefore, the volume of the differential section is —

$$dV = l h dx = 4\sqrt{(R^2 - x^2)(r^2 - x^2)} dx$$

whence

$$V = 8 \int_0^r \sqrt{(R^2 - x^2)(r^2 - x^2)} dx$$

Upon substituting the given values of  $R$  and  $r$  in this formula, it becomes

$$V = 8 \int_0^{\frac{1}{2}} \sqrt{(4 - x^2)(\frac{1}{4} - x^2)} dx$$

This expression belongs to the class of functions known as elliptic integrals, and therefore cannot be integrated directly. Accordingly, we proceed to evaluate  $V$  by mechanical quadrature. For this purpose it will be convenient to put

$$x = \frac{1}{2} \sin \theta$$

whence

$$dx = \frac{1}{2} \cos \theta d\theta$$

and the preceding expression for  $V$  becomes

$$V = \int_0^{\frac{\pi}{2}} \cos^2 \theta \sqrt{16 - \sin^2 \theta} d\theta \quad (367)$$

We now tabulate  $F(\theta) \equiv \omega \cos^2 \theta \sqrt{16 - \sin^2 \theta}$  (where  $\omega = 10^\circ = \pi \div 18$ ) as follows :

$\theta$	$'F$	$F(\theta)$	$\Delta'$	$\Delta''$	$\Delta'''$	$\Delta^{iv}$
— 15°		0.6500		— 371		
— 5	0.0000	0.6927	+ 427	427	— 56	+ 56
+ 5	0.6927	0.6927	0	427	0	56
15	1.3427	0.6500	— 427	371	+ 56	47
25	1.9129	0.5702	798	268	103	31
35	2.3765	0.4636	1066	268	134	+ 12
45	2.7201	0.3436	1200	— 134	146	— 4
55	2.9449	0.2248	1188	+ 12	142	22
65	3.0663	0.1214	1034	154	120	37
75	3.1117	0.0454	760	274	83	37
85	3.1168	0.0051	— 403	357	+ 46	46
95		0.0051	0	403	0	— 46
+ 105		0.0454	+ 403	— 403	— 46	
				+ 357		

Accordingly, we take

$$t = 5^\circ \qquad i = 8 \qquad t + i\omega = 85^\circ$$

and proceed by formula (259): thus, observing that  $\Delta'_{-\frac{1}{2}}$ ,  $\Delta''_{-\frac{1}{2}}$ , . . . . and  $\Delta'_{i+\frac{1}{2}}$ ,  $\Delta''_{i+\frac{1}{2}}$ , . . . . are *all zero*, and remembering that the factor  $\omega$  has already been introduced, we find

$${}^1F_{-\frac{1}{2}} = 0$$

and

$$V = {}^1F_{i+\frac{1}{2}} = 3.1168 \text{ cubic feet}$$

88. Various other problems and applications of a similar nature might be added; indeed, Astronomy itself presents a large variety of such. But the leading principles of our subject have already been developed, explained, and exemplified. We therefore feel confident in leaving the student who has thoroughly mastered these principles, believing him fully capable of solving any further questions or problems that may arise in his practice.

## EXAMPLES.

1. Derive the expression for the sum of the cubes of the first  $r$  integers.

$$\text{Ans. } \frac{1}{4} r^2 (r+1)^2.$$

2. Find from the following ephemeris the instant when *Autumn commences*; that is, the instant when the Sun's right-ascension ( $\alpha$ ) equals twelve hours.

Date 1898	Sun's R.A. $\alpha$	Date 1898	Sun's R.A. $\alpha$
	<sup>h</sup> <sup>m</sup> <sup>s</sup>		<sup>h</sup> <sup>m</sup> <sup>s</sup>
Sept. 13	11 25 47.56	Sept. 25	12 8 54.44
16	11 36 33.99	28	12 19 43.35
19	11 47 20.29	Oct. 1	12 30 34.30
22	11 58 6.94	4	12 41 27.92

$$\text{Ans. Sept. } 22^{\text{d}} 12^{\text{h}} 34^{\text{m}}.8.$$

3. From the ephemeris of the moon's latitude given below, determine the instant of greatest latitude north.

Date 1898	Moon's Latitude	Date 1898	Moon's Latitude
	<sup>°</sup> <sup>'</sup> <sup>"</sup>		<sup>°</sup> <sup>'</sup> <sup>"</sup>
July 9.0	+5 7 9.3	July 10.5	+5 16 48.7
9.5	5 14 28.1	11.0	5 12 9.7
10.0	+5 17 38.3	11.5	+5 3 52.8

$$\text{Ans. July } 10^{\text{d}} 3^{\text{h}} 27^{\text{m}}.4.$$

4. Given the equation

$$\sin(z - 43^\circ) = 0.92 \sin^4 z$$

to determine the root which falls in the second quadrant.

$$\text{Ans. } 101^\circ 17' 43''.$$

5. Given the following table of the longitude of *Mercury's* ascending node ( $\theta$ ):

Year	$\theta$		
	$^{\circ}$	$'$	$''$
1700	44	46	34.42
1800	45	57	39.28
1900	47	8	45.40
2000	48	19	52.78
2100	49	31	1.42

Express  $\theta$  as a function of  $\tau$ ; where  $\tau$  is the elapsed time from 1900, reckoned in terms of one century as the unit.

$$\text{Ans. } \theta = 47^{\circ} 8' 45''.40 + 4266''.75\tau + 0''.630\tau^2.$$

## APPENDIX.

## ON THE SYMBOLIC METHOD OF DEVELOPMENT.

89. While many of the formulae and results in the foregoing text have been derived by somewhat indirect methods, yet the processes employed in every case have involved nothing but purely algebraic operations and principles.

For the benefit of such students as may be interested, we shall now devote a brief space to the more direct and potent form of development known as the *symbolic method*. In this our only purpose is to exhibit the simple manner in which the fundamental formulae of the text may be deduced; leaving the student to enter for himself upon the broader field thus opened by suggestion.

90. Let us define the *symbol of operation*  $\triangle$  by the relation

$$\Delta F(T) = F(T+\omega) - F(T) \quad (368)$$

from which we formulate the following

DEFINITION: The operation of  $\Delta$  upon any function of  $T$  produces the increment in the function which corresponds to the finite increment  $\omega$  in the variable  $T$ .

The relation (368) may be more briefly expressed in the form

$$\Delta F_n = F_{n+1} - F_n = \Delta'_n \quad (369)$$

where  $n$  can have any value. Thus, taking  $n = 0$ , and referring to the schedule on page 15, we have

$$\Delta F_0 = F_1 - F_0 = A'_0 \quad (370)$$

Similarly

$$\left. \begin{aligned} \triangle F_1 &= F_2 - F_1 = A'_1 \\ \triangle F_2 &= F_3 - F_2 = A'_2 \\ . &. . . . . \\ \triangle F_s &= F_{s+1} - F_s = A'_s \end{aligned} \right\} \quad (371)$$

Thus it is evident that the effect of operating with  $\Delta$  upon any *tabular* function is simply to form the *first difference* of that function and the succeeding tabular value. Whence it is evident that we have

$$\left. \begin{aligned} \Delta \Delta F_0 &= \Delta(A'_0) = A''_0 \\ \Delta \Delta F_1 &= \Delta(A'_1) = A''_1 \\ \dots\dots\dots \\ \Delta \Delta F_s &= \Delta(A'_s) = A''_s \end{aligned} \right\}$$

}

(372)

It follows that the operation of  $\Delta\Delta$  upon any tabular function produces the second difference bearing the same subscript. But this double operation of  $\Delta$  may be conveniently characterized by  $\Delta^2$ ; hence we write

$$\Delta^2 F_0 = A''_0 \quad , \quad \Delta^2 F_1 = A''_1 \quad , \quad \dots\dots\dots , \quad \Delta^2 F_s = A''_s$$

(373)

In like manner,  $i$  denoting *any* integer, we have

$$\left. \begin{aligned} \Delta^i F_0 &= \Delta(\Delta^{i-1} F_0) = \Delta(A_0^{(i-1)}) = A_0^{(i)} \\ \Delta^i F_1 &= \Delta(\Delta^{i-1} F_1) = \Delta(A_1^{(i-1)}) = A_1^{(i)} \\ \dots\dots\dots \\ \Delta^i F_s &= \Delta(\Delta^{i-1} F_s) = \Delta(A_s^{(i-1)}) = A_s^{(i)} \end{aligned} \right\}$$

}

(374)

and, more generally,  $n$  being a non-integer,

$$\Delta^i F_n = (\Delta\Delta\Delta \dots\dots i \text{ times}) F_n = A_n^{(i)}$$

(375)

91. Let us now consider the operation of differentiating  $F(T)$  with respect to  $T$  and multiplying the derivative by  $\omega$ . Denoting the operator in this process by  $D$ , we then have

$$D F_n = \omega \frac{d F_n}{d T} = \omega F'_n$$

(376)

also

$$D^2 F_n = D D F_n = \omega \frac{d}{d T} (\omega F'_n) = \omega^2 F''_n$$

(377)

$$D^i F_n = (D D D \dots\dots i \text{ times}) F_n = \left( \omega \frac{d}{d T} \right)^i F_n = \omega^i F_n^{(i)}$$

(378)

92. The fundamental laws or principles governing the combination of symbols of *quantity* in algebraic operations are the following :

I. The *Distributive Law*, by virtue of which

$$a(p+q+r) = ap + aq + ar$$

II. The *Commutative Law*, expressed by the equation

$$ab = ba$$

III. The *Index Law*, which asserts the relation

$$a^r \times a^s = a^{r+s}$$

We proceed to show that the symbols of *operation*,  $\Delta$  and  $D$ , when combined each with itself or with symbols of *quantity* in the manner indicated below, also obey these fundamental laws; and hence that, wherever found in similar combinations,  $\Delta$  and  $D$  may be treated *algebraically precisely as if they were themselves mere symbols of quantity*. We shall first consider the symbol  $\Delta$ .

(1). By definition, we have

$$\begin{aligned} \Delta(F_n + f_n + \dots) &= (F_{n+1} + f_{n+1} + \dots) - (F_n + f_n + \dots) \\ &= (F_{n+1} - F_n) + (f_{n+1} - f_n) + \dots \\ &= \Delta F_n + \Delta f_n + \dots \end{aligned}$$

which proves the *Distributive Law* for the symbol  $\Delta$ .

(2) The factor  $a$  being a constant, we have

$$\Delta a F_n = a F_{n+1} - a F_n = a (F_{n+1} - F_n) = a \Delta F_n$$

thus showing that  $\Delta$  combines with *constant quantities* in accordance with the *Commutative Law*.

(3)  $r$  and  $s$  denoting positive integers, the relation (375) gives

$$\Delta^r \Delta^s F_n = \Delta^r (\Delta^s F_n) = \Delta^r A_n^{(s)} = A_n^{(r+s)} = \Delta^{r+s} F_n$$

or

$$\Delta^r \Delta^s = \Delta^{r+s}$$

Therefore, so far as *positive integral indices* are concerned, the symbol  $\Delta$  obeys the *Index Law*.

93. Retaining the limitations and the notation used above, similar results are easily obtained for the operator  $D$ , as follows:

$$\begin{aligned}
 (1) \quad D(F_n + f_n + \dots) &= \omega \frac{d}{dT} (F_n + f_n + \dots) \\
 &= \omega \frac{dF_n}{dT} + \omega \frac{df_n}{dT} + \dots \\
 &= DF_n + Df_n + \dots \\
 (2) \quad DaF_n &= \left( \omega \frac{d}{dT} \right) aF_n = a\omega \frac{dF_n}{dT} = aDF_n \\
 (3) \quad D^r D^s F_n &= \left( \omega^r \frac{d^r}{dT^r} \right) \left( \omega^s \frac{d^s}{dT^s} \right) F_n = \omega^r \omega^s \left( \frac{d^r}{dT^r} \right) \left( \frac{d^s}{dT^s} \right) F_n \\
 &= \omega^{r+s} \left( \frac{d^{r+s}}{dT^{r+s}} \right) F_n = D^{r+s} F_n
 \end{aligned}$$

These relations prove that — within the limitations imposed — the symbol  $D$  obeys the fundamental laws of algebraic combination.

94. To a limited extent it is necessary to consider negative powers of  $\Delta$  and  $D$ . Now the meaning and use of  $\Delta^{-1}$ ,  $\Delta^{-2}$ , . . . , and of  $D^{-1}$ ,  $D^{-2}$ , . . . are easily understood: thus, from the foregoing definitions, we have

$$\Delta('F_n) = F_n$$

where  $'F_n$  is defined as in the schedule on page 134. Then, in analogy with the usual mode of expressing inverse functions, we may write

$$'F_n = \Delta^{-1}F_n$$

Whence we have

$$\Delta \Delta^{-1}F_n = \Delta('F_n) = F_n \quad (379)$$

which shows (1) that the operation of  $\Delta \Delta^{-1} (= \Delta^0)$  leaves the subject function unaltered, and (2) that *negative powers of  $\Delta$  also obey the Index Law*.

The relation

$$\Delta^{-1}F_n = 'F_n \quad (380)$$

may be taken as the definition of the operator  $\Delta^{-1}$ . Similarly, we have

$$\Delta^{-2}F_n = ''F_n, \quad \Delta^{-3}F_n = ''''F_n, \quad \dots \quad (381)$$

Again, consider the relation

$$DF_n = \omega \frac{dF_n}{dT} = v \quad (382)$$

which, from the point of view above taken, may be written

$$F_n = D^{-1}v \quad (383)$$

Then we have

$$DD^{-1}v = DF_n = v \quad (384)$$

whence we see that *negative* powers of  $D$  likewise follow the *Index Law*.

Moreover, from equation (382), we obtain

$$dF_n = \omega^{-1}v dT$$

and therefore

$$F_n = \omega^{-1} \int v dT$$

which, with (383), gives

$$D^{-1}v = \omega^{-1} \int v dT \quad (385)$$

It follows that the operation of  $D^{-1}$  is equivalent to an integration. More specifically : *Operating upon any function with  $D^{-1}$  integrates that function with respect to  $T$  and divides the resulting integral by  $\omega$ .*

In like manner we have

$$D^{-2}F_n = \omega^{-2} \iint F_n dT^2 \quad (386)$$

and so on.

95. Having thus defined and explained the use of the symbols of operation,  $\Delta^{-2}, \Delta^{-1}, \Delta^0, \Delta, \Delta^2, \dots$ , and  $D^{-2}, D^{-1}, D^0, D, D^2, \dots$ ; and having shown that these symbols may in general be combined algebraically as if they were merely symbols of quantity, we now proceed to derive the fundamental relations of the text, as originally proposed.

96. The theorem of the change in sign of the odd orders of differences caused by inverting a given series of functions is easily proved. To this end, let us suppose that  $A_i^{(r)}$ , of the direct or given series, becomes  $[A_i^{(r)}]$  when that series has been inverted. Then, since

$$\Delta F_i = F_{i+1} - F_i = A_i'$$

we have

$$-\Delta F_i = F_i - F_{i+1} = [A_i']$$

Whence, regarding  $-\Delta$  as operator, it follows that

$$(-\Delta)^2 F_i = [A_i''], \quad (-\Delta)^3 F_i = [A_i'''], \quad \dots, \quad (-\Delta)^r F_i = [A_i^{(r)}]$$

and therefore

$$[A_i^{(r)}] = (-\Delta)^r F_i = (-1)^r \Delta^r F_i = (-1)^r A_i^{(r)} \quad (387)$$

which establishes Theorem III.



99. Expressing the relation (388) in logarithmic form, we get

$$D = \log_e(1 + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \quad (392)$$

whence

$$\left. \begin{aligned} D^2 &= \Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 - \frac{5}{6} \Delta^5 + \dots \\ D^3 &= \Delta^3 - \frac{3}{2} \Delta^4 + \frac{7}{4} \Delta^5 - \dots \\ &\dots \end{aligned} \right\} \quad (393)$$

From these relations the formulae (45)—or the equivalent group (165)—immediately follow.

100. We next consider the question of reducing the tabular interval from  $\omega$  to  $m\omega$ , as discussed in §19. Since in the preceding definitions of  $\Delta$  and  $D$  the magnitude of the interval is arbitrary, we have here only to denote by  $\partial$  and  $d$  the corresponding symbols in the reduced series; evidently the same relations will then exist between  $\partial$  and  $d$  as were found above for  $\Delta$  and  $D$ . Thus we obtain

$$d = m\omega \frac{d}{dT} = m \left( \omega \frac{d}{dT} \right) = mD \quad (394)$$

and since, by (388), we have

$$1 + \Delta = e^D$$

we must have also

$$1 + \partial = e^d = e^{mD} \quad (395)$$

Whence we find

$$1 + \partial = (1 + \Delta)^m = 1 + m\Delta + \frac{m(m-1)}{2} \Delta^2 + \frac{m(m-1)(m-2)}{6} \Delta^3 + \dots$$

and therefore

$$\left. \begin{aligned} \partial &= m\Delta + \frac{m(m-1)}{2} \Delta^2 + \frac{m(m-1)(m-2)}{6} \Delta^3 + \dots \\ \partial^2 &= m^2 \Delta^2 + m^2(m-1) \Delta^3 + \dots \\ \partial^3 &= m^3 \Delta^3 + \dots \\ &\dots \end{aligned} \right\} \quad (396)$$

which are equivalent to the relations expressed in (64).

101. The equation

$$\Delta F_0 = F_1 - F_0$$

may be written in the form

$$(1 + \Delta) F_0 = F_1 \quad (397)$$

Hence the binomial  $1 + \Delta$  may be defined as an operator whose effect is to raise by unity the subscript of the subject function. Whence we have

$$\left. \begin{aligned} (1 + \Delta)^2 F_0 &= (1 + \Delta) F_1 = F_2 \\ (1 + \Delta)^3 F_0 &= (1 + \Delta) F_2 = F_3 \end{aligned} \right\} \quad (398)$$

and generally

$$(1 + \Delta)^n F_0 = F_n \quad (399)$$

We therefore obtain

$$F_n = (1 + \Delta)^n F_0 = \left( 1 + n\Delta + \frac{n(n-1)}{2!} \Delta^2 + \frac{n(n-1)(n-2)}{3!} \Delta^3 + \dots \right) F_0$$

or

$$F_n = F_0 + nA'_0 + \frac{n(n-1)}{2!} A''_0 + \frac{n(n-1)(n-2)}{3!} A'''_0 + \dots \quad (400)$$

which is the fundamental formula of interpolation due to NEWTON.

102. We now find it convenient to introduce a new symbol of operation, which, from its similarity and relation to  $\Delta$ , we shall designate  $\nabla$ : this operator is defined by the equation

$$\nabla F_i = F_i - F_{i-1} = A'_{i-1} \quad (401)$$

From this relation we at once derive

$$\left. \begin{aligned} \nabla^2 F_i &= \nabla A'_{i-1} = A''_{i-2} \\ \nabla^3 F_i &= \nabla A''_{i-2} = A'''_{i-3} \\ \nabla^4 F_i &= \nabla A'''_{i-3} = A^{iv}_{i-4} \\ &\dots \dots \dots \end{aligned} \right\} \quad (402)$$

whence it appears that the operation of  $\nabla^r$  upon any tabular function produces the difference of order  $r$  which falls upon the *upward inclined diagonal* through that function; whereas the successive operations of  $\Delta$  produce, as already shown, those differences falling upon the *downward* diagonal line. Moreover, from the complete similarity of character of these two operators, it is obvious that  $\nabla$  likewise follows the fundamental laws of algebraic combination.

The relation between  $\nabla$  and  $\Delta$  is easily found: thus, from (401), we obtain

$$(1 - \nabla) F_i = F_{i-1} \quad (403)$$

also, from (397), we have

$$(1 + \Delta) F_{i-1} = F_i \quad (404)$$

Whence we find

$$(1+\Delta)(1-\nabla)F_i = (1+\Delta)F_{i-1} = F_i$$

and therefore

$$1-\nabla = (1+\Delta)^{-1} \quad (405)$$

which gives

$$\log(1-\nabla) = -\log(1+\Delta) \quad (406)$$

Again, combining (388) and (405), we obtain

$$1-\nabla = e^{-\Delta} \quad (407)$$

103. As an immediate application of the preceding relations, let us derive the formula (75). By means of (388), equation (399) becomes

$$F_n = (1+\Delta)^n F_0 = e^{n\Delta} F_0$$

whence, changing the sign of  $n$ , we find

$$\begin{aligned} F_{-n} &= e^{-n\Delta} F_0 = (e^{-\Delta})^n F_0 = (1-\nabla)^n F_0 \\ &= (1-n\nabla + \frac{n(n-1)}{2!} \nabla^2 - \frac{n(n-1)(n-2)}{3!} \nabla^3 + \dots) F_0 \end{aligned}$$

Therefore

$$F_{-n} = F_0 - nA'_{-1} + \frac{n(n-1)}{2!} A''_{-2} - \frac{n(n-1)(n-2)}{3!} A'''_{-3} + \dots \quad (408)$$

which is NEWTON'S Formula for *backward* interpolation, as given by (75).

104. Formula (66) of the text is easily deduced by means of the identity

$$\Delta = (1+\Delta) - 1$$

Thus we find

$$\begin{aligned} \Delta^i F_0 &= \{(1+\Delta)-1\}^i F_0 \\ &= \left\{ (1+\Delta)^i - i(1+\Delta)^{i-1} + \frac{i(i-1)}{2!} (1+\Delta)^{i-2} - \dots \right\} F_0 \end{aligned}$$

whence, by (399), we obtain

$$A_0^{(i)} = F_i - iF_{i-1} + \frac{i(i-1)}{2!} F_{i-2} - \frac{i(i-1)(i-2)}{3!} F_{i-3} + \dots \quad (409)$$

which is the same as equation (66).

105. We now pass to the derivation of the fundamental formulae of mechanical quadrature. Since  $D = \log(1 + \Delta)$ , we have

$$\begin{aligned} D^{-1}F_n &= \{\log(1 + \Delta)\}^{-1}F_n = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right)^{-1}F_n \\ &= (\Delta^{-1} + \frac{1}{2} - \frac{1}{12}\Delta + \frac{1}{24}\Delta^2 - \frac{1}{720}\Delta^3 + \frac{1}{160}\Delta^4 - \frac{8}{60480}\Delta^5 + \dots)F_n \end{aligned}$$

Whence, interpreting the first member according to (385), and the term  $\Delta^{-1}F_n$  as in (380), we find

$$\omega^{-1} \int F_n dT = {}^1F_n + \frac{1}{2}F_n - \frac{1}{12}A'_n + \frac{1}{24}A''_n - \frac{1}{720}A'''_n + \frac{1}{160}A^{iv}_n - \frac{8}{60480}A^v_n + \dots \quad (410)$$

This is the fundamental relation of quadrature, from which the formula (a) of (250) is at once derived. To obtain (b) of (250) involving the differences  $A'_{n-1}$ ,  $A''_{n-2}$ ,  $A'''_{n-3}$ , . . . , we have only to employ the relation (406), and the above development becomes

$$\begin{aligned} D^{-1}F_n &= \{\log(1 + \Delta)\}^{-1}F_n = \{-\log(1 - \nabla)\}^{-1}F_n \\ &= \left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + \dots\right)^{-1}F_n \\ &= (\nabla^{-1} - \frac{1}{2} - \frac{1}{12}\nabla - \frac{1}{24}\nabla^2 - \frac{1}{720}\nabla^3 - \frac{1}{160}\nabla^4 - \frac{8}{60480}\nabla^5 - \dots)F_n \end{aligned}$$

the interpretation of which gives

$$\omega^{-1} \int F_n dT = {}^1F_{n+1} - \frac{1}{2}F_n - \frac{1}{12}A'_{n-1} - \frac{1}{24}A''_{n-2} - \frac{1}{720}A'''_{n-3} - \frac{1}{160}A^{iv}_{n-4} - \frac{8}{60480}A^v_{n-5} - \dots \quad (411)$$

agreeing with formula (b) of (250).

106. Similarly, we obtain for the second integration

$$\begin{aligned} D^{-2}F_n &= \{\log(1 + \Delta)\}^{-2}F_n = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right)^{-2}F_n \\ &= (\Delta^{-2} + \Delta^{-1} + \frac{1}{12} - \frac{1}{240}\Delta^2 + \frac{1}{240}\Delta^3 - \frac{2}{60480}\Delta^4 + \frac{1}{60480}\Delta^5 - \dots)F_n \end{aligned}$$

Now the first pair of terms in the right-hand member may be written

$$(\Delta^{-2} + \Delta^{-1})F_n = \Delta^{-2}(1 + \Delta)F_n = \Delta^{-2}F_{n+1} = {}^{II}F_{n+1}$$

and therefore the preceding expression becomes

$$\omega^{-2} \iint F_n dT^2 = {}^{II}F_{n+1} + \frac{1}{12}F_n - \frac{1}{240}A''_n + \frac{1}{240}A'''_n - \frac{2}{60480}A^{iv}_n + \frac{1}{60480}A^v_n - \dots \quad (412)$$

from which (324) immediately follows.

Again, we find

$$\begin{aligned}
 D^{-2} F_n &= \{\log(1+\Delta)\}^{-2} F_n = \{-\log(1-\nabla)\}^{-2} F_n \\
 &= \left( \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + \dots \right)^{-2} F_n \\
 &= (\nabla^{-2} - \nabla^{-1} + \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 4} \nabla^2 - \frac{1}{2 \cdot 4 \cdot 6} \nabla^3 - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6} \nabla^4 - \frac{1 \cdot 9}{6 \cdot 6 \cdot 4 \cdot 8} \nabla^5 - \dots) F_n \quad (413)
 \end{aligned}$$

Transforming the first two terms of the last expression, we obtain

$$(\nabla^{-2} - \nabla^{-1}) F_n = \nabla^{-2} (1 - \nabla) F_n = \nabla^{-2} (1 + \Delta)^{-1} F_n$$

Now, because the operation of  $1 + \Delta$  raises by unity the subscript of the subject function (§101), it follows that the operation of  $(1 + \Delta)^{-1}$  *diminishes* that subscript by one unit. Accordingly, we have

$$(\nabla^{-2} - \nabla^{-1}) F_n = \nabla^{-2} (1 + \Delta)^{-1} F_n = \nabla^{-2} F_{n-1} = {}''F_{n+1}$$

and hence the relation (413) gives

$$\omega^{-2} \iint F_n dT^2 = {}''F_{n+1} + \frac{1}{1 \cdot 2} F_n - \frac{1}{2 \cdot 4 \cdot 6} A''_{n-2} - \frac{1}{2 \cdot 4 \cdot 6} A'''_{n-3} - \frac{2 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 4 \cdot 8 \cdot 6} A^{iv}_{n-4} - \frac{1 \cdot 9}{6 \cdot 6 \cdot 4 \cdot 8} A^v_{n-5} - \dots \quad (414)$$

which is equivalent to the formula (326). These expressions complete the fundamental relations of mechanical quadrature.



# TABLES.

TABLE I. — NEWTON'S INTERPOLATING COEFFICIENTS.

BINOMIAL COEFFICIENTS FOR																	
Interval	n	J''				J'''				J''''				Jv			
		J''		J'''		J''''		J''''		Jv							
		$\frac{n(n-1)}{2}$	Dif.	$\frac{n(n-1)(n-2)}{6}$	Dif.	$\frac{n(n-1)(n-2)(n-3)}{24}$	Dif.	$\frac{n(n-1)(n-2)(n-3)(n-4)}{120}$	Dif.								
0.00	0.00	.00000	-495	.00000	+328	.00000	-245	.00000	+196	.00000	-120	.00000	+111	.00000	-62	.00000	+39
0.01	.01	-.00495	485	+.00328	319	+.00196	-237	+.00196	188	+.00196	120	+.00196	103	+.00196	57	+.00196	34
0.02	.02	.00980	475	.00647	308	.00384	-227	.00384	179	.00384	120	.00384	96	.00384	51	.00384	31
0.03	.03	.01455	465	.00955	299	.00563	-219	.00563	172	.00563	120	.00563	89	.00563	46	.00563	26
0.04	.04	.01920	455	.01254	290	.00735	-211	.00735	164	.00735	120	.00735	82	.00735	40	.00735	22
0.05	.05	.02375	445	.01544	280	.00899	-201	.00899	157	.00899	120	.00899	75	.00899	36	.00899	18
0.06	.06	-.02820	435	+.01824	270	+.01056	-194	+.01056	150	+.01056	120	+.01056	68	+.01056	30	+.01056	14
0.07	.07	.03255	425	.02094	261	.01206	-185	.01206	142	.01206	120	.01206	61	.01206	26	.01206	11
0.08	.08	.03680	415	.02355	252	.01348	-178	.01348	135	.01348	120	.01348	54	.01348	21	.01348	7
0.09	.09	.04095	405	.02607	243	.01483	-169	.01483	129	.01483	120	.01483	48	.01483	16	.01483	4
0.10	.10	.04500	395	.02850	234	.01612	-162	.01612	121	.01612	120	.01612	42	.01612	11	.01612	0
0.11	.11	-.04895	385	+.03084	225	+.01733	-154	+.01733	116	+.01733	120	+.01733	35	+.01733	8	+.01733	-3
0.12	.12	.05280	375	.03309	216	.01849	-147	.01849	109	.01849	120	.01849	28	.01849	-3	.01849	6
0.13	.13	.05655	365	.03525	207	.01958	-140	.01958	102	.01958	120	.01958	23	.01958	2	.01958	10
0.14	.14	.06020	355	.03732	199	.02060	-132	.02060	97	.02060	120	.02060	16	.02060	5	.02060	12
0.15	.15	.06375	345	.03931	191	.02157	-125	.02157	90	.02157	120	.02157	10	.02157	9	.02157	15
0.16	.16	-.06720	335	+.04122	182	+.02247	-119	+.02247	85	+.02247	120	+.02247	5	+.02247	13	+.02247	18
0.17	.17	.07055	325	.04304	173	.02332	-111	.02332	80	.02332	120	.02332	2	.02332	17	.02332	20
0.18	.18	.07380	315	.04477	166	.02412	-105	.02412	73	.02412	120	.02412	7	.02412	21	.02412	23
0.19	.19	.07695	305	.04643	157	.02485	-99	.02485	69	.02485	120	.02485	12	.02485	24	.02485	25
0.20	.20	.08000	295	.04800	149	.02554	-92	.02554	63	.02554	120	.02554	18	.02554	27	.02554	28
0.21	.21	-.08295	285	+.04949	142	+.02617	-86	+.02617	58	+.02617	120	+.02617	24	+.02617	31	+.02617	30
0.22	.22	.08580	275	.05091	133	.02675	-80	.02675	53	.02675	120	.02675	29	.02675	34	.02675	32
0.23	.23	.08855	265	.05224	126	.02728	-74	.02728	48	.02728	120	.02728	34	.02728	38	.02728	34
0.24	.24	.09120	-255	.05350	+119	.02776	-68	.02776	44	.02776	120	.02776	39	.02776	+40	.02776	-36
0.25	0.25	-.09375		+.05469		+.02820		+.02820		+.02820	120	+.02820		+.02820		+.02820	



TABLE II.—STIRLING'S INTERPOLATING COEFFICIENTS.

STIRLING'S COEFFICIENTS FOR										STIRLING'S COEFFICIENTS FOR									
Interval										Interval									
Interval	$n$	$J''$		$J'''$		$J^{iv}$		$J^v$		Interval	$n$	$J''$		$J'''$		$J^{iv}$		$J^v$	
		$\frac{n^2}{2}$	Diff.	$\frac{n(n^2-1)}{6}$	Diff.	$\frac{n^2(n^2-1)}{24}$	Diff.	$\frac{n(n^2-1)(n^2-4)}{120}$	Diff.			$\frac{n^2}{2}$	Diff.	$\frac{n(n^2-1)}{6}$	Diff.	$\frac{n^2(n^2-1)(n^2-4)}{120}$	Diff.		
0.00	0.00	.00000	+ 5	.00000	-167	.00000	.00000	.00000	+ 33	0.25	0.25	+.03125	+255	-.03906	-134	-.00244	-19	+.00769	+25
.01	.01	+.00005	15	-.00167	166	.00000	+.00033	+.00033	34	.26	.26	.03380	265	.04040	132	.00263	19	.00794	25
.02	.02	.00020	25	.00333	167	-.00002	.00067	.00067	33	.27	.27	.03645	275	.04172	129	.00282	19	.00819	24
.03	.03	.00045	35	.00500	166	.00004	.00100	.00100	33	.28	.28	.03920	285	.04301	126	.00301	20	.00843	24
.04	.04	.00080	45	.00666	165	.00007	.00133	.00133	33	.29	.29	.04205	295	.04427	123	.00321	20	.00867	23
.05	.05	.00125	55	.00831	165	.00010	.00166	.00166	33	.30	.30	.04500	305	.04550	120	.00341	21	.00890	22
.06	.06	+.00180	65	-.00996	165	-.00015	+.00199	+.00199	33	.31	.31	+.04805	315	-.04670	117	-.00362	21	+.00912	21
.07	.07	.00245	75	.01161	164	.00020	.00232	.00232	33	.32	.32	.05120	325	.04787	114	.00383	21	.00933	21
.08	.08	.00320	85	.01325	163	.00026	.00265	.00265	32	.33	.33	.05445	335	.04901	111	.00404	22	.00954	19
.09	.09	.00405	95	.01488	162	.00033	.00297	.00297	32	.34	.34	.05780	345	.05012	107	.00426	22	.00973	19
.10	.10	.00500	105	.01650	161	.00041	.00329	.00329	32	.35	.35	.06125	355	.05119	103	.00448	22	.00992	19
.11	.11	+.00605	115	-.01811	160	-.00050	+.00361	+.00361	32	.36	.36	+.06480	365	-.05222	100	-.00470	22	+.01011	17
.12	.12	.00720	125	.01971	159	.00059	.00393	.00393	31	.37	.37	.06845	375	.05322	97	.00492	23	.01028	17
.13	.13	.00845	135	.02130	158	.00069	.00424	.00424	31	.38	.38	.07220	385	.05419	92	.00515	22	.01045	15
.14	.14	.00980	145	.02288	156	.00080	.00455	.00455	31	.39	.39	.07605	395	.05511	89	.00537	23	.01060	15
.15	.15	.01125	155	.02444	154	.00092	.00486	.00486	30	.40	.40	.08000	405	.05600	85	.00560	23	.01075	14
.16	.16	+.01280	165	-.02598	153	-.00104	+.00516	+.00516	30	.41	.41	+.08405	415	-.05685	80	-.00583	22	+.01089	13
.17	.17	.01445	175	.02751	152	.00117	.00546	.00546	30	.42	.42	.08820	425	.05765	77	.00605	23	.01102	12
.18	.18	.01620	185	.02903	149	.00131	.00576	.00576	29	.43	.43	.09245	435	.05842	72	.00628	22	.01114	11
.19	.19	.01805	195	.03052	148	.00145	.00605	.00605	29	.44	.44	.09680	445	.05914	67	.00650	23	.01125	11
.20	.20	.02000	205	.03200	146	.00160	.00634	.00634	28	.45	.45	.10125	455	.05981	63	.00673	22	.01136	9
.21	.21	+.02205	215	-.03346	143	-.00176	+.00662	+.00662	27	.46	.46	+.10580	465	-.06044	59	-.00695	22	+.01145	8
.22	.22	.02420	225	.03489	142	.00192	.00689	.00689	28	.47	.47	.11045	475	.06103	54	.00717	22	.01153	7
.23	.23	.02645	235	.03631	139	.00209	.00717	.00717	26	.48	.48	.11520	485	.06157	49	.00739	21	.01160	7
.24	.24	.02880	245	.03770	136	.00226	.00743	.00743	26	.49	.49	.12005	495	.06206	44	.00760	21	.01167	5
0.25	0.25	+.03125		-.03906		-.00244	+.00769	+.00769	26	0.50	0.50	+.12500		-.06250		-.00781		+.01172	

TABLE II.—STIRLING'S INTERPOLATING COEFFICIENTS. 221

STIRLING'S COEFFICIENTS FOR									
Interval	n	$\Delta''$				$\Delta'''$			
		$\Delta''$		$\Delta'''$		$\Delta'''$		$\Delta'''$	
		$\frac{n^2}{2}$	Dif.	$\frac{n(n^2-1)}{6}$	Dif.	$\frac{n^2(n^2-1)}{24}$	Dif.	$\frac{n^2(n^2-1)(n^2-4)}{120}$	Dif.
Interval	n								
0.50	.50	+ .12500	+505	— .06250	—39	— .00781	—21	+ .01172	+3
.51	.51	.13005	515	.06289	34	.00802	20	.01176	4
.52	.52	.13520	525	.06323	29	.00822	20	.01179	2
.53	.53	.14045	535	.06352	24	.00842	19	.01181	+1
.54	.54	.14580	545	.06376	18	.00861	18	.01182	0
.55	.55	.15125	555	.06394	12	.00879	18	.01182	—1
.56	.56	+ .15680	565	— .06406	7	— .00897	17	+ .01181	2
.57	.57	.16245	575	.06413	—2	.00914	16	.01179	4
.58	.58	.16820	585	.06415	+5	.00930	16	.01175	5
.59	.59	.17405	595	.06410	10	.00946	14	.01170	5
.60	.60	.18000	605	.06400	16	.00960	14	.01165	7
.61	.61	+ .18605	615	— .06384	23	— .00974	12	+ .01158	8
.62	.62	.19220	625	.06361	28	.00986	11	.01150	9
.63	.63	.19845	635	.06333	35	.00997	11	.01141	10
.64	.64	.20480	645	.06298	42	.01008	9	.01131	12
.65	.65	.21125	655	.06256	48	.01017	7	.01119	13
.66	.66	+ .21780	665	— .06208	54	— .01024	7	+ .01106	13
.67	.67	.22445	675	.06154	61	.01031	5	.01093	15
.68	.68	.23120	685	.06093	68	.01036	3	.01078	16
.69	.69	.23805	695	.06025	75	.01039	2	.01062	18
.70	.70	.24500	705	.05950	82	.01041	—1	.01044	18
.71	.71	+ .25205	715	— .05868	89	— .01042	+2	+ .01026	20
.72	.72	.25920	725	.05779	96	.01040	3	.01006	21
.73	.73	.26645	735	.05683	103	.01037	5	.00985	22
.74	.74	.27380	745	.05580	+111	.01032	+7	.00963	—23
0.75	.75	+ .28125	+745	— .05469	—	— .01025	—	+ .00940	—
STIRLING'S COEFFICIENTS FOR									
Interval	n	$\Delta''$				$\Delta'''$			
		$\Delta''$		$\Delta'''$		$\Delta'''$		$\Delta'''$	
		$\frac{n^2}{2}$	Dif.	$\frac{n(n^2-1)}{6}$	Dif.	$\frac{n^2(n^2-1)}{24}$	Dif.	$\frac{n^2(n^2-1)(n^2-4)}{120}$	Dif.
Interval	n								
0.75	.75	+ .28125	+755	— .05469	+119	— .01025	+8	+ .00940	—24
.76	.76	.28880	765	.05350	126	.01017	11	.00916	26
.77	.77	.29645	775	.05224	133	.01006	13	.00890	27
.78	.78	.30420	785	.05091	142	.00993	16	.00863	28
.79	.79	.31205	795	.04949	149	.00977	17	.00835	29
.80	.80	.32000	805	.04800	157	.00960	20	.00806	30
.81	.81	+ .32805	815	— .04643	166	— .00940	22	+ .00776	31
.82	.82	.33620	825	.04477	173	.00918	25	.00745	33
.83	.83	.34445	835	.04304	182	.00893	27	.00712	33
.84	.84	.35280	845	.04122	191	.00866	31	.00679	35
.85	.85	.36125	855	.03931	199	.00835	33	.00644	36
.86	.86	+ .36980	865	— .03732	207	— .00802	35	+ .00608	36
.87	.87	.37845	875	.03525	216	.00767	39	.00572	38
.88	.88	.38720	885	.03309	225	.00728	42	.00534	39
.89	.89	.39605	895	.03084	234	.00686	45	.00495	40
.90	.90	.40500	905	.02850	243	.00641	48	.00455	42
.91	.91	+ .41405	915	— .02607	252	— .00593	51	+ .00413	42
.92	.92	.42320	925	.02355	261	.00542	55	.00371	43
.93	.93	.43245	935	.02094	270	.00487	58	.00328	44
.94	.94	.44180	945	.01824	280	.00429	62	.00284	45
.95	.95	.45125	955	.01544	290	.00367	66	.00239	46
.96	.96	+ .46080	965	— .01254	299	— .00301	69	+ .00193	47
.97	.97	.47045	975	.00955	308	.00232	74	.00146	48
.98	.98	.48020	985	.00647	319	.00158	77	.00098	48
.99	.99	.49005	+995	— .00328	+328	— .00081	+81	+ .00050	—50
1.00	1.00	+ .50000		.00000		.00000		.00000	

TABLE III. — BESSEL'S INTERPOLATING COEFFICIENTS.

BESSEL'S COEFFICIENTS FOR									
Interval	n	J''				J'''			
		J''		J'''		J'''		J''	
		$\frac{n(n-1)}{2}$	Diff.	$\frac{n(n-1)(n-2)}{6}$	Diff.	$\frac{n(n-1)(n-2)}{6}$	Diff.	$\frac{(n+1) \dots (n-2)}{24}$	Diff.
0.00	.00	.00000	-495	.00000	+81	.00000	+83	.00000	+8
.01	.01	-.00495	485	+.00081	76	-.00008	82	+.00083	8
.02	.02	.00980	475	.00157	71	.00016	81	.00165	7
.03	.03	.01455	465	.00228	66	.00023	80	.00246	7
.04	.04	.01920	455	.00294	62	.00030	79	.00326	6
.05	.05	.02375	445	.00356	58	.00036	78	.00405	7
.06	.06	-.02820	435	+.00414	53	-.00043	77	+.00483	5
.07	.07	.03255	425	.00467	48	.00048	76	.00560	5
.08	.08	.03680	415	.00515	45	.00053	74	.00636	5
.09	.09	.04095	405	.00560	40	.00058	74	.00710	5
.10	.10	.04500	395	.00600	36	.00063	72	.00784	4
.11	.11	-.04895	385	+.00636	33	-.00067	70	+.00856	3
.12	.12	.05280	375	.00669	28	.00070	70	.00926	4
.13	.13	.05655	365	.00697	25	.00074	68	.00996	3
.14	.14	.06020	355	.00722	22	.00077	66	.01064	2
.15	.15	.06375	345	.00744	18	.00079	65	.01130	2
.16	.16	-.06720	335	+.00762	14	-.00081	64	+.01195	2
.17	.17	.07055	325	.00776	11	.00083	62	.01259	2
.18	.18	.07380	315	.00787	8	.00085	60	.01321	2
.19	.19	.07695	305	.00795	5	.00086	59	.01381	-1
.20	.20	.08000	295	.00800	+2	.00086	57	.01440	0
.21	.21	-.08295	285	+.00802	-1	-.00087	56	+.01497	-1
.22	.22	.08580	275	.00801	4	.00087	54	.01553	0
.23	.23	.08855	265	.00797	7	.00087	52	.01607	0
.24	.24	.09120	-255	.00790	-9	.00086	+50	.01659	+1
.25	.25	-.09375		+.00781		-.00085		+.01709	+1

BESSEL'S COEFFICIENTS FOR									
Interval	n	J''				J'''			
		J''		J'''		J'''		J''	
		$\frac{n(n-1)}{2}$	Diff.	$\frac{n(n-1)(n-2)}{6}$	Diff.	$\frac{n(n-1)(n-2)}{6}$	Diff.	$\frac{(n+1) \dots (n-2)}{24}$	Diff.
0.25	.25	-.09375	-245	+.00781	-11	-.00085	+49	+.01709	+1
.26	.26	.09620	235	.00770	14	.00084	46	.01758	1
.27	.27	.09855	225	.00756	17	.00083	45	.01804	2
.28	.28	.10080	215	.00739	18	.00081	43	.01849	2
.29	.29	.10295	205	.00721	21	.00079	42	.01892	2
.30	.30	.10500	195	.00700	23	.00077	39	.01934	2
.31	.31	-.10695	185	+.00677	24	-.00075	38	+.01973	3
.32	.32	.10880	175	.00653	27	.00072	35	.02011	2
.33	.33	.11055	165	.00626	28	.00070	34	.02046	3
.34	.34	.11220	155	.00598	29	.00067	31	.02080	4
.35	.35	.11375	145	.00569	31	.00063	30	.02111	3
.36	.36	-.11520	135	+.00538	33	-.00060	28	+.02141	4
.37	.37	.11655	125	.00505	34	.00056	26	.02169	3
.38	.38	.11780	115	.00471	35	.00053	23	.02195	4
.39	.39	.11895	105	.00436	36	.00049	22	.02218	4
.40	.40	.12000	95	.00400	37	.00045	20	.02240	4
.41	.41	-.12095	85	+.00363	38	-.00041	17	+.02260	5
.42	.42	.12180	75	.00325	39	.00036	16	.02277	4
.43	.43	.12255	65	.00286	40	.00032	13	.02293	4
.44	.44	.12320	55	.00246	40	.00028	12	.02306	5
.45	.45	.12375	45	.00206	40	.00023	9	.02318	4
.46	.46	-.12420	35	+.00166	41	-.00019	7	+.02327	5
.47	.47	.12455	25	.00125	42	.00014	6	.02334	5
.48	.48	.12480	15	.00083	41	.00009	3	.02340	4
.49	.49	.12495	-5	+.00042	-42	-.00005	+1	.02343	+5
.50	.50	-.12500		.00000		.00000		+.02344	

TABLE III. — BESSEL'S INTERPOLATING COEFFICIENTS.

Interval		BESSEL'S COEFFICIENTS FOR									
		$\Delta''$					$\Delta'''$				
		$\frac{n(n-1)}{2}$	Dif.	$\frac{n(n-1)}{2}$	Dif.	$\frac{n(n-1)}{2}$	$\frac{n(n-1)(n-2)}{6}$	Dif.	$\frac{n(n-1)(n-2)}{6}$	Dif.	$\Delta^v$
$n$	$n$										
0.50	0.75	— .12500	+ 5	— .09375	+255	— .00781	— 9	— .01709	—50	— .00085	+1
.51	.76	.12495	15	.09120	265	.00790	7	.01659	52	.00086	+1
.52	.77	.12480	25	.08855	275	.00797	4	.01607	54	.00087	0
.53	.78	.12455	35	.08580	285	.00801	— 1	.01553	56	.00087	0
.54	.79	.12420	45	.08295	295	.00802	+ 2	.01497	57	.00087	— 1
.55	.80	.12375	55	.08000	305	.00800	5	.01440	59	.00086	0
.56	.81	— .12320	65	— .07695	315	— .00795	8	+ .01381	60	+ .00086	— 1
.57	.82	.12255	75	.07380	325	.00787	11	.01321	62	.00085	2
.58	.83	.12180	85	.07055	335	.00776	14	.01259	64	.00083	2
.59	.84	.12095	95	.06720	345	.00762	18	.01195	65	.00081	2
.60	.85	.12000	105	.06375	355	.00744	22	.01130	66	.00079	2
.61	.86	— .11895	115	— .06020	365	— .00722	25	+ .01064	68	+ .00077	3
.62	.87	.11780	125	.05655	375	.00697	28	.00996	70	.00074	4
.63	.88	.11655	135	.05280	385	.00669	33	.00926	70	.00070	3
.64	.89	.11520	145	.04895	395	.00636	36	.00856	72	.00067	4
.65	.90	.11375	155	.04500	405	.00600	40	.00784	74	.00063	5
.66	.91	— .11220	165	— .04095	415	— .00560	45	+ .00710	74	+ .00058	5
.67	.92	.11055	175	.03680	425	.00515	48	.00636	76	.00053	5
.68	.93	.10880	185	.03255	435	.00467	53	.00560	77	.00048	5
.69	.94	.10695	195	.02820	445	.00414	58	.00483	78	.00043	7
.70	.95	.10500	205	.02375	455	.00356	62	.00405	79	.00036	6
.71	.96	— .10295	215	— .01920	465	— .00294	66	+ .00326	80	+ .00030	7
.72	.97	.10080	225	.01455	475	.00228	71	.00246	81	.00023	7
.73	.98	.09855	235	.00980	485	.00157	76	.00165	82	.00016	8
.74	.99	.09620	+245	.00495	+495	— .00081	+81	+ .00083	—83	+ .00008	— 8
0.75	1.00	— .09375		.00000		.00000		.00000		.00000	

TABLE IV. — NEWTON'S COEFFICIENTS FOR  $F'(T)$ .

Interval		COEFFICIENTS FOR									
		$J''$					$J'''$				
		$n - \frac{1}{2}$	$\frac{n^2}{2} - n + \frac{1}{3}$	$\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4}$	Diff.	$\frac{n^4}{24} - \frac{n^3}{3} + \dots + \frac{1}{3}$	$J^{iv}$	Diff.	$\frac{n^5}{120} - \frac{n^4}{24} + \frac{n^3}{6} - \frac{n^2}{4} + \frac{n}{2} - \frac{1}{6}$	Diff.	$J^v$
0.00	0.00	-0.50	+ .33333	- .25000	+909	+ .20000	- .06510	-745	- .06510	+566	+ .04131
.01	.01	.49	.32338	.24091	894	.19175	.05944	735	.05944	555	.03682
.02	.02	.48	.31353	.23197	880	.18368	.05389	725	.05389	542	.03245
.03	.03	.47	.30378	.22317	865	.17578	.04847	715	.04847	529	.02821
.04	.04	.46	.29413	.21452	850	.16805	.04318	705	.04318	518	.02409
.05	.05	.45	.28458	.20602	836	.16048	.03800	695	.03800	506	.02009
.06	.06	.44	.27513	.19766	821	.15308	.03294	685	.03294	493	+ .01621
.07	.07	.43	.26578	.18945	807	.14584	.02801	675	.02801	482	.01245
.08	.08	.42	.25653	.18138	793	.13876	.02319	665	.02319	471	.00880
.09	.09	.41	.24738	.17345	778	.13185	.01848	655	.01848	458	.00527
.10	.10	.40	.23833	.16567	765	.12509	.01390	645	.01390	448	+ .00185
.11	.11	.39	.22938	.15802	751	.11848	.00942	635	.00942	435	- .00145
.12	.12	.38	.22053	.15051	737	.11203	.00507	625	.00507	425	.00465
.13	.13	.37	.21178	.14314	723	.10573	.00082	615	.00082	413	.00774
.14	.14	.36	.20313	.13591	710	.09958	+ .00331	605	+ .00331	402	.01072
.15	.15	.35	.19458	.12881	696	.09358	.00733	595	.00733	392	.01360
.16	.16	.34	.18613	.12185	683	.08773	+ .00738	585	+ .00738	380	- .01638
.17	.17	.33	.17778	.11502	669	.08202	+ .00153	575	+ .00153	369	.01905
.18	.18	.32	.16953	.10833	656	.07645	- .00422	565	- .00422	359	.02162
.19	.19	.31	.16138	.10177	644	.07102	.00987	555	.00987	348	.02410
.20	.20	.30	.15333	.09533	630	.06573	.01542	545	.01542	338	.02648
.21	.21	.29	.14538	.08903	617	.06058	- .02087	535	- .02087	327	- .02876
.22	.22	.28	.13753	.08286	605	.05556	.03246	525	.03246	317	.03095
.23	.23	.27	.12978	.07681	591	.05068	.03147	515	.03147	307	.03305
.24	.24	.26	.12213	.07090	580	.04593	.03662	505	.03662	297	.03506
.25	.25	-0.25	+ .11458	- .06510	+580	+ .04131	- .04167	-505	- .04167	+297	- .03698

TABLE IV.—NEWTON'S COEFFICIENTS FOR  $F'(T)$ .

COEFFICIENTS FOR										COEFFICIENTS FOR									
Interval					Interval					Interval					Interval				
$n$					$n$					$n$					$n$				
$\Delta''$					$\Delta''$					$\Delta''$					$\Delta''$				
$n - \frac{1}{2}$					$n - \frac{1}{2}$					$n - \frac{1}{2}$					$n - \frac{1}{2}$				
$\frac{n^2}{2} - n + \frac{1}{3}$					$\frac{n^2}{2} - n + \frac{1}{3}$					$\frac{n^2}{2} - n + \frac{1}{3}$					$\frac{n^2}{2} - n + \frac{1}{3}$				
Diff.					Diff.					Diff.					Diff.				
$\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4}$					$\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4}$					$\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4}$					$\frac{n^3}{6} - \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4}$				
Diff.					Diff.					Diff.					Diff.				
$\frac{n^4}{24} - \frac{n^3}{3} + \dots + \frac{1}{5}$					$\frac{n^4}{24} - \frac{n^3}{3} + \dots + \frac{1}{5}$					$\frac{n^4}{24} - \frac{n^3}{3} + \dots + \frac{1}{5}$					$\frac{n^4}{24} - \frac{n^3}{3} + \dots + \frac{1}{5}$				
Diff.					Diff.					Diff.					Diff.				
$n$					$n$					$n$					$n$				
0.50	0.00	-.04167	-495	+04167	0.50	0.00	-.04167	-495	+04167	0.50	0.00	-.04167	-495	+04167	0.50	0.00	-.04167	-495	+04167
.51	+0.01	.04662	485	.04453	.51	+0.01	.04662	485	.04453	.51	+0.01	.04662	485	.04453	.51	+0.01	.04662	485	.04453
.52	.02	.05147	475	.04730	.52	.02	.05147	475	.04730	.52	.02	.05147	475	.04730	.52	.02	.05147	475	.04730
.53	.03	.05622	465	.04997	.53	.03	.05622	465	.04997	.53	.03	.05622	465	.04997	.53	.03	.05622	465	.04997
.54	.04	.06087	455	.05254	.54	.04	.06087	455	.05254	.54	.04	.06087	455	.05254	.54	.04	.06087	455	.05254
.55	.05	.06542	445	.05502	.55	.05	.06542	445	.05502	.55	.05	.06542	445	.05502	.55	.05	.06542	445	.05502
.56	.06	-.06987	435	+.05740	.56	.06	-.06987	435	+.05740	.56	.06	-.06987	435	+.05740	.56	.06	-.06987	435	+.05740
.57	.07	.07422	425	.05969	.57	.07	.07422	425	.05969	.57	.07	.07422	425	.05969	.57	.07	.07422	425	.05969
.58	.08	.07847	415	.06189	.58	.08	.07847	415	.06189	.58	.08	.07847	415	.06189	.58	.08	.07847	415	.06189
.59	.09	.08262	405	.06399	.59	.09	.08262	405	.06399	.59	.09	.08262	405	.06399	.59	.09	.08262	405	.06399
.60	.10	.08667	395	.06600	.60	.10	.08667	395	.06600	.60	.10	.08667	395	.06600	.60	.10	.08667	395	.06600
.61	.11	-.09062	385	+.06792	.61	.11	-.09062	385	+.06792	.61	.11	-.09062	385	+.06792	.61	.11	-.09062	385	+.06792
.62	.12	.09447	375	.06975	.62	.12	.09447	375	.06975	.62	.12	.09447	375	.06975	.62	.12	.09447	375	.06975
.63	.13	.09822	365	.07150	.63	.13	.09822	365	.07150	.63	.13	.09822	365	.07150	.63	.13	.09822	365	.07150
.64	.14	.10187	355	.07316	.64	.14	.10187	355	.07316	.64	.14	.10187	355	.07316	.64	.14	.10187	355	.07316
.65	.15	.10542	345	.07473	.65	.15	.10542	345	.07473	.65	.15	.10542	345	.07473	.65	.15	.10542	345	.07473
.66	.16	-.10887	335	+.07622	.66	.16	-.10887	335	+.07622	.66	.16	-.10887	335	+.07622	.66	.16	-.10887	335	+.07622
.67	.17	.11222	325	.07762	.67	.17	.11222	325	.07762	.67	.17	.11222	325	.07762	.67	.17	.11222	325	.07762
.68	.18	.11547	315	.07894	.68	.18	.11547	315	.07894	.68	.18	.11547	315	.07894	.68	.18	.11547	315	.07894
.69	.19	.11862	305	.08018	.69	.19	.11862	305	.08018	.69	.19	.11862	305	.08018	.69	.19	.11862	305	.08018
.70	.20	.12167	295	.08133	.70	.20	.12167	295	.08133	.70	.20	.12167	295	.08133	.70	.20	.12167	295	.08133
.71	.21	-.12462	285	+.08241	.71	.21	-.12462	285	+.08241	.71	.21	-.12462	285	+.08241	.71	.21	-.12462	285	+.08241
.72	.22	.12747	275	.08341	.72	.22	.12747	275	.08341	.72	.22	.12747	275	.08341	.72	.22	.12747	275	.08341
.73	.23	.13022	265	.08433	.73	.23	.13022	265	.08433	.73	.23	.13022	265	.08433	.73	.23	.13022	265	.08433
.74	.24	.13287	255	.08517	.74	.24	.13287	255	.08517	.74	.24	.13287	255	.08517	.74	.24	.13287	255	.08517
.75	+0.25	-.13542	245	+.08594	.75	+0.25	-.13542	245	+.08594	.75	+0.25	-.13542	245	+.08594	.75	+0.25	-.13542	245	+.08594

TABLE V. — STIRLING'S COEFFICIENTS FOR  $F'(T)$ .Note: — The coefficient for  $J'' =$  the given argument  $n$ .

COEFFICIENTS FOR				COEFFICIENTS FOR									
Interval	n	J'''		J''		Interval	n	J'''		J''			
		$\frac{n^2}{2} - \frac{1}{6}$	Diff.	$\frac{n^3}{6} - \frac{n}{12}$	Diff.			$\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{30}$	Diff.	$\frac{n^3}{6} - \frac{n}{12}$	Diff.	$\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{30}$	Diff.
0.00		— .16667	+ 5	.00000	— 83	0.25		— .13542	+ 255	— .01823	— 51	+ .02568	— 61
.01		.16662	15	— .00083	84	.26		.13287	265	.01874	48	.02507	63
.02		.16647	25	.00167	83	.27		.13022	275	.01922	45	.02444	65
.03		.16622	35	.00250	82	.28		.12747	285	.01967	43	.02379	67
.04		.16587	45	.00332	83	.29		.12462	295	.02010	40	.02312	70
.05		.16542	55	.00415	81	.30		.12167	305	.02050	37	.02242	71
.06		— .16487	65	— .00496	82	.31		— .11862	315	— .02087	34	+ .02171	74
.07		.16422	75	.00578	80	.32		.11547	325	.02121	30	.02097	76
.08		.16347	85	.00658	80	.33		.11222	335	.02151	27	.02021	77
.09		.16262	95	.00738	79	.34		.10887	345	.02178	24	.01944	79
.10		.16167	105	.00817	77	.35		.10542	355	.02202	20	.01865	82
.11		— .16062	115	— .00894	77	.36		— .10187	365	— .02222	17	+ .01783	83
.12		.15947	125	.00971	76	.37		.09822	375	.02239	13	.01700	85
.13		.15822	135	.01047	74	.38		.09447	385	.02252	9	.01615	87
.14		.15687	145	.01121	73	.39		.09062	395	.02261	6	.01528	88
.15		.15542	155	.01194	71	.40		.08667	405	.02267	— 1	.01440	90
.16		— .15387	165	— .01265	70	.41		— .08262	415	— .02268	+ 3	+ .01350	92
.17		.15222	175	.01335	68	.42		.07847	425	.02265	7	.01258	93
.18		.15047	185	.01403	66	.43		.07422	435	.02258	11	.01165	95
.19		.14862	195	.01469	64	.44		.06987	445	.02247	16	.01070	97
.20		.14667	205	.01533	63	.45		.06542	455	.02231	20	.00973	98
.21		— .14462	215	— .01596	60	.46		— .06087	465	— .02211	25	+ .00875	100
.22		.14247	225	.01656	58	.47		.05622	475	.02186	29	.00775	100
.23		.14022	235	.01714	56	.48		.05147	485	.02157	34	.00675	108
.24		.13787	+ 245	.01770	— 53	.49		.04662	+ 495	.02123	+ 40	.00572	— 103
0.25		— .13542		— .01823		0.50		— .04167		— .02083		+ .00469	

Interval	COEFFICIENTS FOR						Interval	COEFFICIENTS FOR					
	$\Delta'''$			$\Delta''$				$\Delta'''$			$\Delta''$		
	$\frac{n^2}{2} - \frac{1}{6}$	Diff.	$\frac{n^3}{6} - \frac{n}{12}$	Diff.	$\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{30}$	Diff.		$\frac{n^2}{2} - \frac{1}{6}$	Diff.	$\frac{n^3}{6} - \frac{n}{12}$	Diff.	$\frac{n^4}{24} - \frac{n^2}{8} + \frac{1}{30}$	Diff.
$n$							$n$						
0.50	— .04167	+505	— .02083	+44	+ .00469	—105	0.75	+ .11458	+755	+ .00781	+202	— .02380	—117
.51	.03662	515	.02039	49	.00364	103	.76	.12213	765	.00983	209	.02497	116
.52	.03147	525	.01990	55	.00258	107	.77	.12978	775	.01192	217	.02613	116
.53	.02622	535	.01935	59	.00151	108	.78	.13753	785	.01409	225	.02729	116
.54	.02087	545	.01876	66	+ .00043	110	.79	.14538	795	.01634	233	.02845	115
.55	.01542	555	.01810	70	— .00067	110	.80	.15333	805	.01867	240	.02960	114
.56	— .00987	565	— .01740	77	— .00177	111	.81	+ .16138	815	+ .02107	249	— .03074	114
.57	— .00422	575	.01663	82	.00288	112	.82	.16953	825	.02356	257	.03188	112
.58	+ .00153	585	.01581	87	.00400	113	.83	.17778	835	.02613	265	.03300	112
.59	.00738	595	.01494	94	.00513	114	.84	.18613	845	.02878	274	.03412	111
.60	.01333	605	.01400	100	.00627	114	.85	.19458	855	.03152	282	.03523	109
.61	+ .01938	615	— .01300	105	— .00741	115	.86	+ .20313	865	+ .03434	291	— .03632	109
.62	.02553	625	.01195	112	.00856	116	.87	.21178	875	.03725	300	.03741	107
.63	.03178	635	.01083	119	.00972	116	.88	.22053	885	.04025	308	.03848	106
.64	.03813	645	.00964	124	.01088	116	.89	.22938	895	.04333	317	.03954	104
.65	.04458	655	.00840	132	.01204	117	.90	.23833	905	.04650	326	.04058	103
.66	+ .05113	665	— .00708	137	— .01321	117	.91	+ .24738	915	+ .04976	335	— .04161	101
.67	.05778	675	.00571	145	.01438	118	.92	.25653	925	.05311	345	.04262	99
.68	.06453	685	.00426	151	.01556	117	.93	.26578	935	.05656	354	.04361	98
.69	.07138	695	.00275	158	.01673	118	.94	.27513	945	.06010	363	.04459	95
.70	.07833	705	— .00117	166	.01791	118	.95	.28458	955	.06373	373	.04554	94
.71	+ .08538	715	+ .00049	172	— .01909	118	.96	+ .29413	965	+ .06746	382	— .04648	91
.72	.09253	725	.00221	179	.02027	118	.97	.30378	975	.07128	392	.04739	89
.73	.09978	735	.00400	187	.02145	117	.98	.31353	985	.07520	402	.04828	87
.74	.10713	+745	.00587	+194	.02262	—118	.99	.32338	+995	.07922	+411	.04915	—85
0.75	+ .11458		+ .00781		— .02380		1.00	+ .33333		+ .08333		— .05000	

Note: — The coefficient for  $\mathcal{A}'' =$  the given argument  $n$ .

TABLE VI. — BESSEL'S COEFFICIENTS FOR  $F'(T)$ .

Interval		COEFFICIENTS FOR									
		$J''$		$J'''$		$J^{iv}$		$J^v$		$J^v$	
		$n - \frac{1}{2}$	$\frac{n^2}{2} - \frac{n}{2} + \frac{1}{12}$	Diff.	$\frac{n^3}{6} - \frac{n^2}{4} + \frac{n}{12} - \frac{1}{120}$	Diff.	$\frac{n^4}{24} - \frac{n^3}{12} + \frac{n^2}{24} - \frac{n}{120}$	Diff.	$\frac{n^5}{120} - \frac{n^4}{24} + \frac{n^3}{24} - \frac{n^2}{24} + \frac{n}{120}$	Diff.	$\frac{n^6}{720} - \frac{n^5}{120} + \frac{n^4}{24} - \frac{n^3}{24} + \frac{n^2}{24} - \frac{n}{120}$
0.00	.00	-0.50	+ .08333	-495	+ .08333	-85	-.00833	+41	-.00833	-178	+ .00094
.01	.01	.49	.07838	485	.08248	91	.00792	42	.00792	181	.00123
.02	.02	.48	.07353	475	.08157	96	.00750	41	.00750	183	.00150
.03	.03	.47	.06878	465	.08061	100	.00709	42	.00709	185	.00176
.04	.04	.46	.06413	455	.07961	105	.00667	41	.00667	188	.00201
.05	.05	.45	.05958	445	.07856	109	.00626	41	.00626	189	.00225
.06	.06	.44	.05513	435	.07747	114	-.00585	41	-.00585	191	+ .00249
.07	.07	.43	.05078	425	.07633	118	.00544	40	.00544	193	.00271
.08	.08	.42	.04653	415	.07515	122	.00504	40	.00504	195	.00292
.09	.09	.41	.04238	405	.07393	126	.00464	39	.00464	196	.00311
.10	.10	.40	.03833	395	.07267	131	.00425	40	.00425	198	.00330
.11	.11	.39	.03438	385	.07136	134	-.00385	38	-.00385	199	+ .00348
.12	.12	.38	.03053	375	.07002	138	.00347	38	.00347	201	.00364
.13	.13	.37	.02678	365	.06864	142	.00309	38	.00309	202	.00380
.14	.14	.36	.02313	355	.06722	145	.00271	37	.00271	202	.00394
.15	.15	.35	.01958	345	.06577	149	.00234	36	.00234	204	.00407
.16	.16	.34	.01613	335	.06428	152	-.00198	36	-.00198	205	+ .00418
.17	.17	.33	.01278	325	.06276	155	.00162	34	.00162	205	.00429
.18	.18	.32	.00953	315	.06121	159	.00128	34	.00128	207	.00438
.19	.19	.31	.00638	305	.05962	162	.00093	33	.00093	206	.00446
.20	.20	.30	.00333	295	.05800	165	.00060	33	.00060	208	.00453
.21	.21	.29	.00038	285	.05635	168	-.00027	31	-.00027	207	+ .00459
.22	.22	.28	-.00247	275	.05467	170	.00004	31	.00004	208	.00463
.23	.23	.27	.00522	265	.05297	173	.00035	30	.00035	209	.00466
.24	.24	.26	.00787	255	.05124	176	.00065	29	.00065	208	.00468
0.25	0.25	-0.25	-.01042	-255	.04948	-176	+ .00094	+29	+ .00094	-208	+ .00469

TABLE VI. — BESSEL'S COEFFICIENTS FOR  $F'(T)$ .

COEFFICIENTS FOR				COEFFICIENTS FOR				Interval			
$\Delta''$				$\Delta'''$				$\Delta''$			
$n - \frac{1}{2}$				$\frac{n^2}{2} - \frac{n}{2} + \frac{1}{12}$				$\frac{n^2}{2} - \frac{n}{2} + \frac{1}{12}$			
$n$				$\Delta'''$				$\Delta'''$			
$n$				$\Delta'''$				$\Delta'''$			
$n$				$\Delta'''$				$\Delta'''$			
$n$				$\Delta'''$				$\Delta'''$			
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$n$				$\Delta'''$				$\Delta'''$			
$n$				$\Delta'''$				$\Delta'''$			
$n$				$\Delta'''$				$\Delta'''$			

The tabular quantities are in units of the sixth decimal.

x	VALUES OF ARGUMENT K.																					x
	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19	0.20	
0.000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.000
.001	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.001
.002	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.002
.003	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	.003
.004	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	.004
.005	0	0	0	0	0	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	.005
.006	0	0	0	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	3	4	.006
.007	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	.007
.008	0	0	1	1	1	2	2	2	3	3	3	4	4	4	4	5	5	5	6	6	6	.008
.009	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6	6	6	7	7	8	8	.009
.010	0	0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	8	9	9	10	10	.010
.011	0	1	1	2	2	3	4	4	5	5	6	7	7	8	8	9	10	10	11	11	12	.011
.012	0	1	1	2	3	4	4	5	6	6	7	8	9	9	10	11	12	12	13	14	14	.012
.013	0	1	2	3	3	4	5	6	7	8	8	9	10	11	12	13	14	14	15	16	17	.013
.014	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	.014
.015	0	1	2	3	4	6	7	8	9	10	11	12	14	15	16	17	18	19	20	21	23	.015
.016	0	1	3	4	5	6	8	9	10	12	13	14	15	17	18	19	20	22	23	24	26	.016
.017	0	1	3	4	6	7	9	10	12	13	14	16	17	19	20	22	23	25	26	27	29	.017
.018	0	2	3	5	6	8	10	11	13	15	16	18	19	21	23	24	26	28	29	31	32	.018
.019	0	2	4	5	7	9	11	13	14	16	18	20	22	23	25	27	29	31	32	34	36	.019
.020	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	.020
.021	0	2	4	7	9	11	13	15	18	20	22	24	26	29	31	33	35	37	40	42	44	.021
.022	0	2	5	7	10	12	15	17	19	22	24	27	29	31	34	36	39	41	44	46	48	.022
.023	0	3	5	8	11	13	16	19	21	24	26	29	32	34	37	40	42	45	48	50	53	.023
.024	0	3	6	9	12	14	17	20	23	26	29	32	35	37	40	43	46	49	52	55	58	.024
0.025	0	3	6	9	13	16	19	22	25	28	31	34	38	41	44	47	50	53	56	59	63	0.025
0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19	0.20		

GIVING  $y$ : TO BE USED IN FINDING  $n$  WHEN  $F_n$  IS GIVEN.

NOTE.— The quantity  $y$  has the same sign as argument  $K$ .

x	VALUES OF ARGUMENT K.																					x
	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19	0.20	
0.025	0	3	6	9	13	16	19	22	25	28	31	34	38	41	44	47	50	53	56	59	63	0.025
.026	0	3	7	10	14	17	20	24	27	30	34	37	41	44	47	51	54	57	61	64	68	.026
.027	0	4	7	11	15	18	22	26	29	33	36	40	44	47	51	55	58	62	66	69	73	.027
.028	0	4	8	12	16	20	24	27	31	35	39	43	47	51	55	59	63	67	71	74	78	.028
.029	0	4	8	13	17	21	25	29	34	38	42	46	50	55	59	63	67	71	76	80	84	.029
.030	0	5	9	14	18	23	27	32	36	41	45	50	54	58	63	68	72	76	81	86	90	.030
.031	0	5	10	14	19	24	29	34	38	43	48	53	58	62	67	72	77	82	86	91	96	.031
.032	0	5	10	15	20	26	31	36	41	46	51	56	61	67	72	77	82	87	92	97	102	.032
.033	0	5	11	16	22	27	33	38	44	49	54	60	65	71	76	82	87	93	98	103	109	.033
.034	0	6	12	17	23	29	35	40	46	52	58	64	69	75	81	87	92	98	104	110	116	.034
.035	0	6	12	18	25	31	37	43	49	55	61	67	74	80	86	92	98	104	110	116	123	.035
.036	0	6	13	19	26	32	39	45	52	58	65	71	78	84	91	97	104	110	117	123	130	.036
.037	0	7	14	21	27	34	41	48	55	62	68	75	82	89	96	103	110	116	123	130	137	.037
.038	0	7	14	22	29	36	43	51	58	65	72	79	87	94	101	108	116	123	130	137	144	.038
.039	0	8	15	23	30	38	46	53	61	68	76	84	91	99	106	114	122	129	137	144	152	.039
.040	0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	.040
.041	0	8	17	25	34	42	50	59	67	76	84	92	101	109	118	126	134	143	151	160	168	.041
.042	0	9	18	26	35	44	53	62	71	79	88	97	106	115	123	132	141	150	159	168	176	.042
.043	0	9	18	28	37	46	55	65	74	83	92	102	111	120	129	139	148	157	166	176	185	.043
.044	0	10	19	29	39	48	58	68	77	87	97	106	116	126	136	145	155	165	174	184	194	.044
.045	0	10	20	30	40	51	61	71	81	91	101	111	122	132	142	152	162	172	182	192	203	.045
.046	0	11	21	32	42	53	63	74	85	95	106	116	127	138	148	159	169	180	190	201	212	.046
.047	0	11	22	33	44	55	66	77	88	99	110	121	133	144	155	166	177	188	199	210	221	.047
.048	0	12	23	35	46	58	69	81	92	104	115	127	138	150	161	173	184	196	207	219	230	.048
.049	0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	.049
0.050	0	12	25	38	50	62	75	88	100	112	125	138	150	162	175	188	200	212	225	238	250	0.050
	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19	0.20	

The tabular quantities are in units of the sixth decimal.

GIVING  $y$ : TO BE USED IN FINDING  $n$  WHEN  $F_n$  IS GIVEN.NOTE. — The quantity  $y$  has the same sign as argument  $K$ .

## COEFFICIENTS FOR COMPUTING

$$F_n = F_0 + n\omega(F'_0 + \frac{n}{2}\alpha + B\beta_0 + \Gamma\gamma).$$

$n$	$B \equiv \frac{n^2}{6}$	Dif.	$\Gamma \equiv \frac{n}{12} \left( \frac{n^2}{2} - 1 \right)$	Dif.
0.00	0.0000		0.0000	
.02	+ .0001	+ 1	— .0017	—17
.04	.0003	2	.0033	16
.06	.0006	3	.0050	17
.08	.0011	5	.0066	16
.10	.0017	6	.0083	17
		7		16
.12	+ .0024		— .0099	
.14	.0033	9	.0116	17
.16	.0043	10	.0132	16
.18	.0054	11	.0148	16
.20	.0067	13	.0163	15
		14		16
.22	+ .0081		— .0179	
.24	.0096	15	.0194	15
.26	.0113	17	.0209	15
.28	.0131	18	.0224	15
.30	.0150	19	.0239	15
		21		14
.32	+ .0171		— .0253	
.34	.0193	22	.0267	14
.36	.0216	23	.0281	14
.38	.0241	25	.0294	13
.40	.0267	26	.0307	13
		27		12
.42	+ .0294		— .0319	
.44	.0323	29	.0331	12
.46	.0353	30	.0343	12
.48	.0384	31	.0354	11
.50	.0417	33	.0365	11
		34		10
.52	+ .0451		— .0375	
.54	.0486	35	.0384	9
.56	.0523	37	.0393	9
.58	.0561	38	.0402	9
0.60	+0.0600	+39	—0.0410	— 8

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